QP CODE: 22000354

22000354

Reg No : ....

Name : ....

MSc DEGREE (CSS) EXAMINATION , JANUARY 2022

Second Semester

CORE - ME010205 - MEASURE AND INTEGRATION

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

81B3C335

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define measurability of a set. Prove that the translate of a measurable set is measurable.
2. State and prove the excision property of Lebesgue measurable sets.
3.
   1. When will you say a property holds almost everyehere on a measurable set E?
   2. Let $\{E_k\}_{k=1}^{\infty}$ is a countable collection of Lebesgue measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then prove that almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.
4. Let f and g are Lebesgue measurable function on E. Prove that the sum $f + g$ is Lebesgue measurable on E.
5. Prove that bounded Lebesgue measurable functions on a set of finite measure E , are Lebesgue integrable over E.
6. Prove that for a non negative Lebesgue measurable function f on E, $\int_E f = 0$ if and only if $f = 0$ a.e. on E.
7. Define a null set with respect to a signed measure. Prove that a set of measure zero with respect to a signed measure need not be a null set.
8. Let $(X, \mathcal{M})$ be a measurable space where $\mathcal{M} = \{X, \phi\}$. Which all are the functions that are measurable with respect to $\mathcal{M}$? Justify?
9. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative measurable function on X. Then prove that there is an increasing seguence $\{\psi_n\}$ of simple functions on X that converges pointwise on X to f and $\lim_{n\to\infty} \int_X \psi_n \, d\mu = \int_X f \, d\mu$

10. Define measurable rectangle, semiring and premeasure.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let I be a non-empty interval, prove that $m^*(I) = l(I)$, the length of I.

12. Is the image of a measurable set under a continuous function measurable? Justify your answer.

13. State and Simple Approximation Theorem

14. Prove that Lebesgue integration of simple functions defined on a set of finite measure E satisfies the properties of Linearity and Monotonicity

15. Let f be a nonnegative Lebesgue measurable function on R. For each Lebesgue measurable subset E of R, define $\mu(E) = \int_E f$, the Lebesgue integral of f over E. Show that μ is a measure on the $\sigma-$ algebra of Lebesgue measurable subsets of R.

16. If $E = \cup_{k=1}^{\infty} E_k$ where each E_k is measurable, prove that E is also measurable.

17. Let $(X, \mathcal{M}, \mu)$ be a measure space and the function f be integrable over X. Then prove the following statements.

    1. If $\{X_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of X whose union is X, then $\int\limits_X f\, d\mu = \lim\limits_{n\to\infty} \int\limits_{X_n} f\, d\mu$
    2. If $\{X_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of X, then $\int\limits_{\bigcap\limits_{n=1}^{\infty} X_n} f\, d\mu = \lim\limits_{n\to\infty} \int\limits_{X_n} f\, d\mu$

18. Let $(X, \mathcal{M}, \mu)$ be a measure space and the function f be integrable over X. Then prove that for each $\epsilon > 0$, there is a $\delta > 0$ such that for any measurable subset E of X, $\mu(E) < \delta$ implies $\int\limits_E |f|\, d\mu < \epsilon.$

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19.
    1. Let E be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set Λ of real numbers for which the collection of translates of $\{\lambda + E\}_{\lambda\in\Lambda}$ is disjoint. Then prove that $m(E) = 0$.
    2. Prove that there exists a non-measurable set of real numbers.

20. State and prove the properties of Linearity, Monotonicity and Additivity over domains of integration for Lebesgue integration of integrable functions on E

21. (i) State and prove the Jordan decomposition Theorem.

(ii) Prove that the Jordan decomposition of a signed measure is unique.

22. (a) State Radon Nikodym Theorem

(b) State and prove Lebesgue Decomposition theorem

(2×5=10 weightage)