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M.Sc. DEGREE (C.S.S.) EXAMINATION, NOVEMBER 2018

Third Semester

Faculty of Science

Branch I—(A) Mathematics

MT 03 C 13—DIFFERENTIAL GEOMETRY

(2012 Admission onwards)

Time : Three Hours | Maximum Weight : 30

Part A

Answer any **five** *out of eight questions.*
Each question has weight 1.

1. Define a vector field and give an example of a vector field, by sketching it.
2. Show by an example that the set of vectors tangent at a point p of a level set might be all of R_p^{n+1}.
3. Define the terms Gauss map and spherical image.
4. Show that for each $a, b, c, d \in R$, the parametrized curve :

   $\alpha(t) = (\cos(at+b), \sin(at+b), ct+d)$ is a geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in R^3.
5. Define a Weingarten map. Why it is called the shape operator ?
6. State Freuet formulas for a plane curve.
7. Define a quadratic form associated with a self adjoint linear transformation $L : V \to V$. When it is said to be positive and negative definite.
8. Define a parametrized n-surface. Give example.

$(5 \times 1 = 5)$

Part B

Answer any **five** *questions out of eight.*
Each question has weight 2.

9. Sketch the typical level curves and the graph of the function $f(x_1, x_2) = x_1^2 - x_2^2$.
10. Let U be an open set in R^{n+1} and let $f : U \to R$ be smooth. Let $p \in U$ be a regular point of f, and $c = f^{-1}(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

Turn over

11. Describe the spherical image when $u = 1$ and $u = 2$ of the given n surface oriented by $\nabla f / \| \nabla f \|$, where f is function defined on the left side of the equation $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = r^2, r > 0$.

12. Let $\alpha : [0, \pi] \to S^2$ be the half great circle in S^2, running from the north pole $p = (0, 0, 1)$ to the south pole $q = (0, 0, -1)$ defined by $\alpha(t)(\sin t, 0, \cos t)$. Show that for $v = (p, v, v_2, 0) \in S_p^2$, $p_\alpha(v) = (q, -v_1, v_2, 0)$.

13. Compute $\nabla_v X$, where $v \in R_p^{n+1}, p \in R^{n+}$ and X given by $X(x_1, x_2) = (x_1, x_2, x_1 x_2, x_2^2)$, $v = (1, 0, 0, 1)$, $n = 1$.

14. Find the length of the connected oriented plane curve $f^{-1}(c)$, oriented by $\nabla f / \| \nabla f \|$, where $f : U \to R$ and c are given by $f(x_1, x_2) = 5x_1 + 12x_2$, $U = \{(x_1, x_2) : x_1^2 + x_2^2 < 169\}$, $c = 0$.

15. On each compact oriented n-surface S in R^{n+1} prove that there exists a point p such that the second fundamental form at p is definite.

16. Find the Gaussian curvature of the parametrized 2-surface :

$$\phi(\theta, \phi) = (a \cos\theta \sin\phi, a \sin\theta \sin\phi, a \cos\phi).$$

(5 × 2 = 10)

Part C

Answer any **three** *out of six questions.*
Each question has weight 5.

17. (a) Show that the graph of any function $f : R^n \to R$ is a level set for same function $F : R^{n+1} \to R$.

    (b) State and prove the existence and uniqueness theorem for integrals curves of smooth tangent vector fields on n-surfaces in R^{n+1}.

18. Let S be a compact connected oriented n-surface in R^{n+1} exhibited as a level set $f^{-1}(c)$ of a smooth function $F : R^{n+1} \to R$ with $\nabla f(p) \neq 0$ for all $p \in S$. Prove that the Gauss map maps S onto the unit sphere S^n.

19. Let C be an oriented plane curve. Prove that there exists a global parametrization of C if and only if C is connected.

20. (a) Prove that the Weingarten map is self adjoint.

(b) Let η be the 1-form on $R^2 - \{0\}$ defined by $\eta = -\frac{x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2$. Prove that for $\alpha : [a, b] \to R^2 - [0]$ any closed piecewise smooth parametrized curve in $R^2 - [0]$, $\int_\alpha \eta = 2\pi k$ for some integer k.

21. (a) Let S be a compact connected oriented n-surface in R^{n+1}. Prove that the Gauss-Kronecker curvature $k(p)$ of S at p is non-zero for all $p \in S$ if and only if the second fundamental form $\Im_p$ of S at p is definite for all $p \in S$.

(b) State and prove inverse function theorem for n-surfaces.

22. (a) Find the Gauss-Kronecker curvature of the parametrized 3-surface ϕ, where :

$$\phi(x, y, z) = \left(x, y, z, x^2 + y^2 + z^2\right).$$

(b) Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self-adjoint linear transformation on V. Show that there exists an orthonormal basis for V consisting of eigen vectors of L.

(3 × 5 = 15)