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Reg. No....................................

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M.Sc. DEGREE (C.S.S.) EXAMINATION, APRIL 2019

Fourth Semester

Faculty of Science

Branch I (A)—Mathematics

MT 04 E01—ANALYTIC NUMBER THEORY

(2012 Admission onwards)

Time : Three Hours — Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Show that Dirichlet multiplication is commutative.
2. Show that Euler totient $\phi(u)$ is multiplicative. Is $\phi(u)$ completely multiplicative. Justify.
3. Show that for $x \geq 1, \sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] = 1$.
4. Define Chebyshev's ψ functions, show that $\psi(x) = \sum_{m \leq \log_2 x} \sum_{P \leq x^{1/m}} \log P$.
5. Show that for $x \geq 2$ $\vartheta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$.
6. Show that if $a \equiv b$ (modus) and d/a and d/m, then d/b.
7. Solve the congruence $5x = 3 \pmod{24}$.
8. If g is a primitive root mod P, where p is an odd prime. Show that $g^2, g^4, \ldots . g^{p-1}$ are quadratic residues mod P.

(5 × 1 = 5)

Turn over

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. If $n \le 1$ show that $\sum_{d/n} \phi(d) = n$.

10. If f and g are multiplicative functions prove that their Dirichlet product $f * g$ is multiplicative.

11. State and prove Euler's summation formula.

12. For any arithmetical function $\propto (n)$ Let A (x) = $\sum_{n \le x} \alpha(n),$ where A (x) = 0 if $x < 1$. Assume f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Prove that

$$\sum_{y < n \le x} \alpha(n) f(n) = \mathrm{A}(x) f(x) - \mathrm{A}(y) f(y) - \int_y^x \mathrm{A}(t) f^1(t)\, dt.$$

13. Prove that if $(a, m) = 1$, the linear congruence $a\, x \equiv b$ (modus) has exactly one solution.

14. State and prove Euler-Fermat theorem.

15. State and prove Wilson's theorem.

16. Let x be an odd integer and $\alpha \ge 3$. Show that $x^{\phi(2^{\alpha})/2} \equiv 1 \left(\text{mod } 2^{\alpha}\right)$.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. If $x \ge 1$, prove the following :

(a) $\sum_{n \le x} \frac{1}{n} = \log x + \mathrm{C} + \mathrm{O}\left(\frac{1}{x}\right).$

(b) $\sum_{n \le x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \int (s) + \mathrm{O}\left(x^{-s}\right)$ if $s > 0, s \ne 1.$

(c) $\sum_{n > x} \frac{1}{n^s} = \mathrm{O}\left(x^{1-s}\right)$ if $s > 1.$

(d) $\sum_{n \le x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + \mathrm{O}\left(x^{\alpha}\right)$ if $a \ge 0.$

18. For all $x \geq 1$, show that $\sum_{n \leq x} d(n) = x \log x + (2C - 1)x + O(\sqrt{x})$, where C is Euler's constant.

19. Let P_n denote the n^{th} prime. Show that the following asymptotic relations are logically equivalent :

(i) $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$

(ii) $\lim_{x \to \infty} \frac{\pi(x) \log \pi(x)}{x} = 1.$

(iii) $\lim_{n \to \infty} \frac{P_n}{n \log n} = 1.$

20. (a) State and prove Chinese remainder theorem.

(b) Prove that the set of lattice points in the plane visible from the origin contains arbitrary large square gaps.

21. (a) Assume $(a, m) = d$ and suppose d/b. Prove that the linear congruence $ax \equiv b \pmod{m}$ has exactly d solutions. Also obtain these solutions.

(b) State and prove Lagrange's theorem for polynomial Congruence.

22. Let P be an odd prime and let d be any positive divisor of P – 1. Prove that in every reduced residue system and P there are exactly $\phi(d)$ numbers a such that $\exp_p(a) = d$.

(3 × 5 = 15)