Distance Spectrum of Graphs

Studies on the distance spectrum of graphs

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By

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Declaration

I, Dr.Indulal G. hereby declare that this report entitled '**Studies on the distance spectrum of graphs**' contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or Institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of this report.

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Dr.Indulal G.

Studies on the distance spectrum of graphs

Contents

Chapter 1

Introduction

The origin of ‘Graph Theory’ dates back to more than two hundred and seventy years when the famous Swiss Mathematician Leonhard Euler (1707 - 1783) solved the ‘Koniğsberg Bridge Problem’ in a talk entitled ‘The solution of a problem relating to the geometry of position’ presented at the St.Petersberg Academy on 26^{th} August, 1735. Since then, the subject has grown both in its theory and its varied applications, initiated by the works of such greats as W.R. Hamilton, De Morgan, A. Cayley and P. J. Heawood. The celebrated ‘4 Color Problem’ which was a major unsolved problem since 1852 and its unique method of solution using computers in 1976 - the first of its kind in Mathematics, also belongs to Graph Theory. In 1874, A. Cayley realized that the problem of finding the number of different paraffines with the formula C_nH_{2n+2} is essentially the same as the problem of counting the number of unrooted trees with n vertices, where no vertex has valency exceeding four. But it was J. J. Sylvester who first used the term ‘graph’ in his celebrated paper ‘Chemistry and Algebra’ in 1877.

The first book on graph theory was written by D. König [7]. Later, C. Berge [8], O. Ore [9] and F. Harary [10] also wrote the first set of books in this subject. N.L. Biggs, E. K. Lloyd and R. J. Wilson [11] has discussed in detail, with the extracts of original work, the growth of graph theory. F. S. Roberts [12] has dealt with a variety of applications of graphs in engineering, technology, biological sciences, archeology, ecology, planning etc. This includes its applications in transportation problems, communication, study of food webs in ecology, round - robin tournament in tennis, the theory of structural balance in sociology etc. In [13], connections of graph theory with other branches of mathematics such as number theory, coding theory are discussed.

The computation of various graph polynomials and the associated spectra have been the topic of many investigations in the recent years. While the problem of computing the characteristic polynomial of adjacency matrix and its spectrum appears to be solved for many large graphs, the related distance polynomial has received much less attention. The idea of distance matrix seems a natural generalization, re ects the structure of the graph in a better way than that of an adjacency matrix . Distance matrix and its spectra have arisen independently from a data communication problem studied by Graham and Pollack in 1971 in which the most important feature is the number of negative eigenvalues of the distance matrix. The distance matrix is more complex than the ordinary adjacency matrix of a graph since the distance matrix is a complete matrix (dense) while the adjacency matrix is more often sparse. Thus the computation of the characteristic polynomial of the distance matrix is computationally a much more difficult problem and, in general, there are no simple analytical solutions except those for a few trees For this reason, distance polynomials of only trees have been studied extensively in the mathematical literature . In This report we study some concepts on distance spectrum of graphs. Recent developments in graph spectra are also available in the spectral graph theory home page, www.sgt.pep.ufrj.br.

Seven research papers are published in international scientific journals under this minor research project. The copies are appended herewith which in turn completes this report.

Chapter 2

Major findings

Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, \ldots, v_p\}$ and size q. The distance matrix or $D-$ matrix, D of G is defined as $D = [d_{ij}]$ where d_{ij} is the distance between v_i and v_j in G. The eigenvalues of the $D-$ matrix of G are called the $D-$ eigenvalues of G and form the $D-$ spectrum of G denoted by $spec_D(G)$. In this chapter we list all the major findings we obtained under this minor research project.

2.1 On distance energy of graphs and a pair of D-equienergetic graphs

Theorem 2.1. *Let G be an $r-$ regular graph of diameter 2 with $spec(G) = \{r, \lambda_2, \ldots\ldots, \lambda_p\}$. Then $spec_D(G) = \{2p - r - 2, -(\lambda_2 + 2), \ldots\ldots, -(\lambda_p + 2)\}$.*

Theorem 2.2. *Let G be an $r-$ regular graph of diameter 1 or 2 with an adjacency matrix A and $spec(G) = \{\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_p\}$. Then $H = G \times K_2$ is $r+1-$ regular and of* *diameter* 2 *or* 3 *with*

$$spec_D(H) = \begin{pmatrix} 5p-2(r+2) & -2\left(\lambda_i+2\right) & -p & 0 \\ 1 & 1 & 1 & p-1 \end{pmatrix}, i = 2 \text{ to } p$$

Theorem 2.3. *The distance energy of the wheel graph is given by* $E_D(W_{1,p}) = 2(p-2+\sqrt{p^2-3p+4})$.

Theorem 2.4. *Let* G *be a* (p,q) *graph of diameter 2 and* μ_1 *be the greatest* $D-$ *eigenvalue. Then* $\mu_1 \geq \dfrac{2p^2-2q-2p}{p}$. *The equality holds if and only if* G *is a regular graph.*

Theorem 2.5. *Let* Δ *be the absolute value of the determinant of the distance matrix* D *of* G. *Then*

$$\sqrt{\left(4p\left(p-1\right)-6q\right)+p\left(p-1\right)\Delta^{2/p}} \leqslant E_D\left(G\right) \leqslant \sqrt{2p\left(2p^2-3q-2p\right)}$$

Theorem 2.6. *Let* G *be a* $r-$ *regular graph of diameter 2. Then*

$$E_D \leqslant 2p-r-2+\sqrt{\left(p-1\right)\left[p\left(r+4\right)-\left(r+2\right)^2\right]}$$

Theorem 2.7. *For any graph* G *of diameter* 2,

$$E_D \leqslant \frac{1}{p}\left\{2p^2-2q-2p+\sqrt{\left(p-1\right)\left[\left(2p+q\right)\left(2p^2-4q\right)-4p^2\right]}\right\}$$

Theorem 2.8. *Let* G *be a connected* $r-$ *regular graph on* p *vertices with* $spec(G) = \{\lambda_1, \lambda_2, \ldots.., \lambda_p\}$. *Then*

$$spec_D(G\nabla G) = \begin{pmatrix} 3p-r-2 & p-r-2 & -\left(\lambda_i+2\right) \\ 1 & 1 & 2 \end{pmatrix}$$

Theorem 2.9. *For every* $p \equiv 0(mod6) \geq 18$, *there exists a pair of equi* $D-$ *energetic regular graphs.*

2.2 On the distance spectra of some graphs

In this section the D-spectra of some graphs and their D-energies are calculated. A pair of D-equienergetic bipartite graphs on $24t$, $t \geq 3$, vertices is constructed.

Theorem 2.10. *Let M be a real symmetric irreducible square matrix of order p in which each row sum is equal to a constant k .Then there exists a polynomial $Q(x)$ such that $Q(M) = J$, where J is the all one square matrix whose order is same as that of M .*

Theorem 2.11. *Let D be the distance matrix of a connected distance regular graph G . Then D is irreducible and there exists a polynomial $P(x)$ such that $P(D) = J$. In this case*

$$P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\cdots(x-\lambda_g)}{(k-\lambda_2)(k-\lambda_3)\ldots(k-\lambda_g)}$$

where k is the unique sum of each row which is also the greatest simple eigenvalue of D , whereas $\lambda_2, \lambda_3, \ldots, \lambda_g$ are the other distinct eigenvalues of D .

Theorem 2.12. *Let G be a graph with distance spectrum $spec_D(G) = \{\mu_1, \mu_2, \ldots, \mu_p\}$. Then*

$$spec_D(D_2G) = \begin{pmatrix} 2\,(\mu_i+1) & -2 \\ 1 & p \end{pmatrix} \ , \ i = 1, 2, \ldots, p \ .$$

Theorem 2.13. $E_D(D_2C_{2n}) = 4n(n+1)$.

Theorem 2.14. *Let G be a distance regular graph with distance regularity k , distance matrix D , and D-spectrum $\{\mu_1 = k, \mu_2, \ldots, \mu_p\}$. Then*

$$spec_D(G \times K_2) = \begin{pmatrix} 2k+p & -p & 2\,\mu_i & 0 \\ 1 & 1 & 1 & p-1 \end{pmatrix} \ , \ i = 2, 3, \ldots, p \ .$$

Theorem 2.15. *Let G be a connected distance regular graph with distance regularity k , distance matrix D , and $spec_D(G) = \{\mu_1 = k, \mu_2, \ldots, \mu_p\}$. Then $spec_D(G \circ K_1)$ consists* *of the numbers*

$$p+k-1+\sqrt{(p+k)^2+(p-1)^2} \quad , \quad p+k-1-\sqrt{(p+k)^2+(p-1)^2}$$

$$\mu_i-1+\sqrt{\mu_i^2+1} \ , \ \mu_i-1-\sqrt{\mu_i^2+1} \quad , \quad i=2,3,\ldots,p \ .$$

Theorem 2.16. *Let G be a connected graph with distance spectrum $spec_D(G)=\{\mu_1=k,\mu_2,\ldots,\mu_p\}$. Then*

$$spec_D(G[K_2])=\begin{pmatrix} 2\mu_i+1 & -1 \\ 1 & p \end{pmatrix} \ , \ i=1,2,\ldots,p \ .$$

Theorem 2.17. *Let G be an r-regular graph of diameter 2 on p vertices with (ordinary) spectrum $\{r,\lambda_2,\ldots,\lambda_p\}$. Then the D-spectrum of the EDC-graph of G consists of the numbers $5p-2r-4$, $2r-p$, $-2\,(\lambda_i+2)$, $i=2,3,\ldots,p$, and $2\,\lambda_i$, $i=2,3,\ldots,p$.*

Theorem 2.18. *There exists a pair of regular non-D-cospectral D-equienergetic bipartite graphs on $24\,t$ vertices, for each $t\geq 3$.*

2.3 Distance spectrum of graph compositions

In this section we obtain the $D-$ spectrum of the cartesian product of two distance regular graphs. The $D-$ spectrum of the lexicographic product $G[H]$ of two graphs G and H when H is regular is also obtained. The $D-$ eigenvalues of the Hamming graphs $Ham(d,n)$ of diameter d and order n^d and those of the C_4 nanotori , T_{k,m,C_4} are determined.

Theorem 2.19. *Let G and H be two distance regular graphs on p and n vertices with distance regularity k and t respectively. Let $spec_D(G) = \{k, \mu_2, \mu_3, ..., \mu_p\}$ and $spec_D(H) = \{t, \eta_2, \eta_3, ..., \eta_n\}$. Then*

$$spec_D(G + H) = \{nk + pt, n\mu_i, p\eta_j, 0\}$$

$i = 2, ..., p$, $j = 2, ..., n$ and 0 is with multiplicity $(p-1)(n-1)$.

Theorem 2.20. *Let $Ham(d,n)$ be the Hamming graph of characteristic n. Then the $D-$ eigenvalues of $Ham(d,n)$ are $dn^{d-1}(n-1)$, 0 and $-n^{d-1}$ with multiplicities 1, $n^d - d(n-1) - 1$ and $d\,(n-1)$ respectively.*

Theorem 2.21. *The distance spectrum of the C_4 nanotori , T_{k,m,C_4} consists of the following numbers*

$$\frac{(m+k)(mk-1)}{4}, -\frac{m}{4}\sec^2\left(\frac{\pi j}{2k}\right), -\frac{m}{4}\operatorname{cosec}^2\left(\frac{\pi r}{2k}\right), -\frac{k}{4}\sec^2\left(\frac{\pi t}{2m}\right), -\frac{k}{4}\operatorname{cosec}^2\left(\frac{\pi l}{2m}\right)$$

where $j \in \{1, 2, ..., k-1\}$ and even, $r \in \{1, 2, ..., k-1\}$ and odd

$t \in \{1, 2, ..., m-1\}$ and even and $l \in \{1, 2, ..., m-1\}$ and odd

together with 0 of multiplicity $(m-1)(k-1)$.

Theorem 2.22. *Let G be a graph with $D-$ matrix D_G and H , an $r-$ regular graph with an adjacency matrix A. Let $spec_D(G) = \{\mu_1, \mu_2, ..., \mu_p\}$ and the ordinary spectrum of*

H be $\{r, \lambda_2, \lambda_3, ..., \lambda_n\}$. *Then*

$$spec_D G[H] = \begin{pmatrix} n\mu_i + 2n - r - 2 & -(\lambda_j + 2) \\ 1 & p \end{pmatrix},\ i = 1\ to\ p\ and\ j = 2\ to\ n-1$$

2.4 $D-$ equienergetic self - complementary graphs

In this section we describe here the distance spectrum of the P_4 join-based self - complementary graphs in the terms of their adjacency spectrum. These results are used to show that there exists $D-$ equienergetic self - complementary graphs of order $p = 48t$ and $24(2t+1),\ t \geq 4$

Theorem 2.23. *Let G be a connected $k-$ regular graph on n vertices, with an adjacency matrix A and spectrum $\{k, \lambda_2, \ldots, \ldots, \lambda_n\}$. Then the distance spectrum of $\mathcal{H}$ consists of $-(\lambda_i + 2)$ and $\lambda_i - 1$, $i = 2, 3, ..., ..., n$, each with multiplicity 2 together with the numbers*

$$\frac{1}{2}\left[7n - 3 \pm \sqrt{(2k+1)^2 + 45n^2 - 12nk - 6n}\right] \ \textit{and}\ -\frac{1}{2}\left[n + 3 \pm \sqrt{(2k+1)^2 + 5n^2 + 4nk + 2n}\right]$$

Theorem 2.24. *For every $n \geq 8$, there exists a pair of $4-$ regular non-cospectral graphs on n vertices.*

Theorem 2.25. *Let G be a connected $4-$ regular graph on n vertices, with an adjacency matrix A and spectrum $\{4, \lambda_2, \ldots, \ldots, \lambda_n\}$. Let $H = L^2(G)$ and $\mathcal{H}$ be the P_4 self-complementary graph obtained from H. Then $E_D(\mathcal{H}) = 3[8(3n-1) + \sqrt{20n^2 + 28n + 49}]$*

Theorem 2.26. *For every $p = 48t$ or $24(2t+1)$, $t \geq 4$, there exists a pair of $D-$ equienergetic self-complementary graphs.*

2.5 Sharp bounds on the distance spectral radius and the distance energy of graphs

In this section we obtain some lower bounds for the distance spectral radius μ_1 and characterize those graphs for which these bounds are best possible. We also obtain an upperbound for $E_D(G)$ and determine those maximal $D-$ energy graphs.

Theorem 2.27. *Let G be a graph with Wiener index W. Then $\mu_1 \geq \dfrac{2W}{p}$ and the equality holds if and only if G is distance regular.*

Theorem 2.28. *Let G be a graph with distance degree sequence $\{D_1, D_2, \ldots, \ldots, D_p\}$. Then*

$$\mu_1 \geqslant \sqrt{\frac{D_1^2 + D_2^2 + D_3^2 + \ldots\ldots.. + D_p^2}{p}}$$

The equality holds if and only if G is distance regular.

Theorem 2.29. *Let G be a graph with distance degree sequence $\{D_1, D_2, \ldots, \ldots, D_p\}$ and second distance degree sequence $\{T_1, T_2, \ldots, \ldots, T_p\}$. Then*

$$\mu_1 \geqslant \sqrt{\frac{T_1^2 + T_2^2 + T_3^2 + \ldots\ldots.. + T_p^2}{D_1^2 + D_2^2 + D_3^2 + \ldots\ldots.. + D_p^2}}$$

Equality holds if and only if G is pseudo distance regular.

Theorem 2.30. *Let G be graph with Wiener index W and distance degree sequence $\{D_1, D_2, \ldots, \ldots, D_p\}$. Then*

$$\mu_1 \geqslant \underset{i}{Max} \frac{1}{p-1}\left((W - D_i) + \sqrt{(W - D_i)^2 + (p-1)\, D_i^2}\right)$$

Theorem 2.31. *With the notations described above*

$$E_D(G) \leqslant \sqrt{\frac{\sum_{i=1}^{p} T_i^2}{\sum_{i=1}^{p} D_i^2}} + (p-1)\sqrt{S - \frac{\sum_{i=1}^{p} T_i^2}{\sum_{i=1}^{p} D_i^2}}$$

where S is the sum of the squares of entries in the distance matrix. Equality holds if and only if either G is a complete graph or a pseudo $k-$ distance regular graph with three distinct $D-$ eigenvalues $\left(k, \sqrt{\frac{S-k^2}{p-1}}, -\sqrt{\frac{S-k^2}{p-1}}\right)$.

2.6 The distance spectrum and energy of the compositions of regular graphs

We describe here the distance spectrum and energy of the join-based compositions of regular graphs in the terms of their adjacency spectrum. These results are used to show that there exists a number of families of sets of noncospectral graphs with equal distance energy, such that for any $n \in \mathbf{N}$, each family contains a set with at least n graphs. The simplest such family consists of sets of complete bipartite graphs.

Theorem 2.32. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues of the adjacency matrix A_{G_i}, $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$. The distance spectrum of $G_1 \nabla G_2$ consists of eigenvalues $-\lambda_{i,j} - 2$ for $i = 1, 2$ and $j = 2, 3, \ldots, n_i$ and two more eigenvalues of the form*

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{\left(n_1 - n_2 - \frac{r_1 - r_2}{2}\right)^2 + n_1 n_2}. \tag{2.1}$$

Theorem 2.33. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices, whose smallest eigenvalue of the adjacency matrix is at least -2 and such that $G_i \ncong K_n$. Then*

$$DE(G_1 \nabla G_2) = 4(n_1 + n_2) - 2(r_1 + r_2) - 8.$$

Theorem 2.34. *For $i = 0, 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$ of the adjacency matrix A_{G_i}. If $r_1 \neq r_2$, then the distance spectrum of $G_0 \nabla (G_1 \cup G_2)$ consists of eigenvalues $-\lambda_{i,j} - 2$ for $i = 0, 1, 2$ and $j = 2, 3, \ldots, n_i$ and three more eigenvalues which are solutions of the cubic equation in ν:*

$$\begin{aligned}&(2n_0 - r_0 - 2 - \nu)(\nu + r_1 + 2)(\nu + r_2 + 2)\\&+[2(\nu + r_0 + 2) - 3n_0][n_1(\nu + r_2 + 2) + n_2(\nu + r_1 + 2)] = 0.\end{aligned} \tag{2.2}$$

Theorem 2.35. *Graphs $K_1\nabla(\mathcal{C}_P \cup G)$, $P \in \mathcal{P}_n$, form a set of DE-equienergetic graphs.*

2.7 Papers Published under this minor research project

1. G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **60**(2008),461–472.
2. G. Indulal, I. Gutman, *On the distance spectra of some graphs*, Math. Commun. **13**(2008), 123 -131.
3. G.Indulal , *Sharp bounds on the distance spectral radius and the distance energy of graphs*, Lin. Algebra Appl. **430** (2009), 106–113.
4. Indulal, G., The distance spectrum of graph compositions, Ars. Mathematica Contemporanea **2** (2009), 93 – 100.
5. Stevanović, D. and Indulal, G., The distance spectrum and energy of the compositions of regular graphs, Appl. Math. Lett. **22** (2009), 136 – 1140.
6. Indulal, G. and Gutman, I., $D-$ equienergetic self - complementary graphs, Kragujevac. J. Math. (2009), 123-131

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ON DISTANCE ENERGY OF GRAPHS

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Abstract

The D-eigenvalues of a graph G are the eigenvalues of its distance matrix D, and the D-energy $E_D(G)$ is the sum of the absolute values of its D-eigenvalues. Two graphs are said to be D-equienergetic if they have the same D-energy. In this note we obtain bounds for the distance spectral radius and D-energy of graphs of diameter 2. Pairs of equiregular D-equienergetic graphs of diameter 2, on $p = 3t + 1$ vertices are also constructed.

INTRODUCTION

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and size (= number of edges) q. The distance matrix or D-matrix, D, of G is defined as

$D = [d_{ij}]$, where d_{ij} is the distance between the vertices v_i and v_j in G. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_p$ of the D-matrix of G are said to be the D-eigenvalues of G and to form the D-spectrum of G, denoted by $spec_D(G)$.

Since the D-matrix of G is symmetric, all of its eigenvalues are real and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. Two graphs G and H are said to be D-cospectral if $spec_D(G) = spec_D(H)$. The D-energy $E_D(G)$ of G is defined as

$$E_D(G) = \sum_{i=1}^{p} |\mu_i| \quad . \tag{1}$$

Eq. (1) is put forward in full analogy to the definition of the (ordinary) graph energy E, namely

$$E(G) = \sum_{i=1}^{p} |\lambda_i| \tag{2}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of the adjacency matrix of G. For basic facts on graph energy E see the book [11]; for the most recent research on E see [10,12,14–16,25,28,29,31,32].

Two graphs with the same D-energy are called D-equienergetic. We are, of course, interested in D-equienergetic graphs only if these are not D-cospectral.

The characteristic polynomial of the D-matrix and the corresponding spectrum were considered in [6–9,13,30]. The D-energy seems to be defined here for the first time.

In this paper we are concerned with the D-spectra and D-energies of graphs of diameter 2. Moore and Moser showed [3] that almost all graphs are of diameter two. Thus a discussion of graphs of small diameter pertains to almost all graphs.

This paper is organized as follows. In the next section we establish the distance spectrum of some graphs of diameter 2 and 3. In the following section a lower bound for the largest eigenvalue of D, and bounds for the D-energy are obtained. In the last section some pairs of equiregular D-equienergetic graphs of diameter 2 are constructed.

All graphs considered in this paper are simple. Our spectral graph theoretic terminology follows that of the book [4].

We shall need the following lemmas.

Lemma 1 [4]. Let G be a graph with an adjacency matrix A and $spec(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Then $\det A = \prod_{i=1}^{p} \lambda_i$. In addition, for any polynomial $P(x)$, $P(\lambda)$ is an eigenvalue of $P(A)$ and hence $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

Lemma 2 [5]. Let

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a symmetric 2×2 block matrix. Then the spectrum of A is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.

Lemma 3 [4]. Let M, N, P, Q be matrices, and let M be invertible. Let

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}.$$

Then $\det S = \det M \cdot \det [Q - PM^{-1}N]$. If M and P commute, then $\det S = \det[MQ - PN]$.

Lemma 4 [4]. Let G be an r-regular connected graph, $r \geq 3$, with $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$. Then

$$spec(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \cdots & \lambda_p + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & p(r-2)/2 \end{pmatrix}.$$

Lemma 5 [4]. Let G be an r-regular connected graph on p vertices with an adjacency matrix A, and let $r, \lambda_2, \ldots, \lambda_m$ be its distinct eigenvalues. Let J be the all-one square matrix of order p. Then there exists a polynomial $P(x)$ such that $P(A) = J$, and

$$P(x) = p \, \frac{(x - \lambda_2)(x - \lambda_3) \cdots (x - \lambda_m)}{(r - \lambda_2)(r - \lambda_3) \cdots (r - \lambda_m)}$$

so that $P(r) = p$ and $P(\lambda_i) = 0$, for all $\lambda_i \neq r$.

Lemma 6 [4,19]. For every $t \geq 3$, there exists a pair of non-cospectral cubic graphs on $2t$ vertices.

THE DISTANCE SPECTRUM OF SOME GRAPHS

In this section we calculate the distance spectrum of some graphs of diameter 2 or 3. The distance energy of some particular graphs are also obtained.

Graphs of diameter 2

Let G be a graph of diameter 2, A its adjacency matrix, and $\overline{A}$ the adjacency matrix of its complement $\overline{G}$. Then $d(u,v)=1$ if u adj v in G, and $d(u,v)=2$ if u adj v in $\overline{G}$. Thus the distance matrix of G is $A+2\overline{A}$.

Lemma 7. Let G be a (p,q)-graph of diameter 2, and let its D-eigenvalues be $\mu_1,\mu_2,\ldots,\mu_p$. Then

$$\sum_{i=1}^{p} \mu_i^2 = 2(2\,p^2-2p-3q)\ .$$

Proof. In the distance matrix D of G there are $2q$ elements equal to unity, and $p(p-1)-2q$ elements equal to two. Therefore,

$$\begin{aligned}\sum_{i=1}^{p} \mu_i^2 &= \sum_{i=1}^{p} (D^2)_{ii} = \sum_{i=1}^{p}\sum_{j=1}^{p} d_{ij}\,d_{ji} = \sum_{i=1}^{p}\sum_{j=1}^{p} (d_{ij})^2 \\ &= (2q)\cdot 1^2 + (p^2-p-2q)\cdot 2^2\end{aligned}$$

and the lemma follows. $\square$

Theorem 1. Let G be an r-regular graph of diameter 2, and let its (ordinary) spectrum be $spec(G)=\{r,\lambda_2,\ldots,\lambda_p\}$. Then the D-spectrum of G is $spec_D(G)=\{2p-r-2,-(\lambda_2+2),\ldots,-(\lambda_p+2)\}$.

Proof. The theorem follows from the fact that the D-matrix of G is $A+2\overline{A}$ and from Lemma 5. $\square$

Examples.

$$spec_D(K_{n,n}) = \begin{pmatrix} 3n-2 & n-2 & -2 \\ 1 & 1 & 2n-2 \end{pmatrix}$$

$$spec_D(CP(n)) = \begin{pmatrix} 2n & -2 & 0 \\ 1 & n & n-1 \end{pmatrix}$$

where $CP(n)$ denotes the $(2n)$-vertex regular graph of degree $2n-2$ (obtained by deleting n independent edges from the complete graph K_{2n}), sometimes referred to as the "cocktail party graph".

The graph product $G \times K_2$

Theorem 2. Let G be an r-regular graph of diameter 1 or 2 with an adjacency matrix A and $spec(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Then $H = G \times K_2$ is $(r+1)$-regular and of diameter 2 or 3 with

$$spec_D(H) = \begin{pmatrix} 5p - 2(r+2) & -2\,(\lambda_i + 2) & -p & 0 \\ 1 & 1 & 1 & p-1 \end{pmatrix}\ , \ i = 2, \ldots, p\ .$$

Proof. Since G is of diameter 1 or 2, its distance matrix is $A + 2\overline{A}$. Then the distance matrix of H is of the form

$$\begin{bmatrix} A + 2\overline{A} & A + 2\overline{A} + J \\ A + 2\overline{A} + J & A + 2\overline{A} \end{bmatrix}.$$

The theorem then follows by Lemma 2. $\square$

The wheel graph $W_{1,p}$ is defined as the join of p-vertex cycle C_p and K_1 [4].

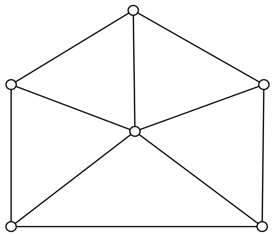

Figure 1: $W_{1,5} = C_5 \nabla K_1$

Theorem 3. The distance energy of the wheel graph is given by $E_D(W_{1,p}) = 2\left(p - 2 + \sqrt{p^2 - 3p + 4}\right)$.

Proof. Let A be an adjacency matrix of C_p with $spec(C_p) = \{2, \lambda_2, \lambda_3, \ldots, \lambda_p\}$. Then the distance matrix of the wheel graph can be written as

$$\begin{bmatrix} A + 2\overline{A} & J_{p\times 1} \\ J_{1\times p} & 0 \end{bmatrix}.$$

By Lemma 3,

$$spec_D\left(W_{1,p}\right) = \begin{pmatrix} p - 2 \pm \sqrt{p^2 - 3p + 4} & -\left(\lambda_i + 2\right) \\ 1 & 1 \end{pmatrix}\ , \ i = 2, \ldots, p\ .$$

Since $\lambda_i + 2 > 0$ for all $i = 2, \ldots, p$, the theorem follows. $\square$

BOUNDS FOR THE SPECTRAL RADIUS AND DISTANCE ENERGY

Theorem 4. Let G be a (p,q)-graph of diameter 2 and μ_1 be its greatest D-eigenvalue. Then $\mu_1 \geq (2\,p^2 - 2q - 2p)/p$. Equality holds if and only if G is a regular graph.

Proof. Let G be a connected graph of diameter 2, and let its vertices be labelled as $v_1, v_2, \ldots, v_p$. Let d_i denote the degree of v_i. Then, as G is of diameter 2, it is easy to observe that the i-th row of D consists of d_i one's and $p - d_i - 1$ two's. Let $x = [1, 1, 1, \ldots, 1]$, the all one vector. Then by the Raleigh Principle

$$\mu_1 \geq \frac{x\,D\,x^T}{x\,x^T} = \frac{1}{p}\sum_{i=1}^{p}(2p - d_i - 2) = \frac{2\,p^2 - 2q - 2p}{p} \, .$$

If G is r-regular, then each row sum of D is equal to $2p - r - 2$ and hence $\mu_1 = 2p - r - 2$ and equality holds. Conversely, if equality holds then x is the eigenvector corresponding to μ_1 and this happens when all row sums of D are equal. Since the i-th row sum is equal to $2p - d_i - 2$, this occurs only when d_i has the same value for all i, i. e., only when G is regular. $\square$

The following theorem gives upper and lower bounds for the energy of graphs of diameter 2.

Theorem 5. Let G be a (p,q)-graph of diameter 2 and let Δ be the absolute value of the determinant of its distance matrix. Then

$$\sqrt{4p\,(p-1) - 6q + p\,(p-1)\,\Delta^{2/p}} \leq E_D(G) \leq \sqrt{2p\,(2\,p^2 - 3q - 2p)} \, .$$

Proof. This proof is fully analogous to what McClelland [24] has done in the case of the ordinary graph energy (see pp. 147-148 in the book [11]). In view of the definition (1) of D-energy and bearing in mind Lemma 7,

$$\begin{aligned} E_D^2 &= \left(\sum_{i=1}^{p}|\mu_i|\right)^2 = \sum_{i=1}^{p}\mu_i^2 + \sum_{i \neq j}|\mu_i|\;|\mu_j| \\ &= 4p\,(p-1) - 6q + \sum_{i \neq j}|\mu_i|\;|\mu_j| \; . \qquad (3) \end{aligned}$$

By using the the inequality between the arithmetic and geometric means we have

$$\frac{1}{p(p-1)} \sum_{i \neq j} |\mu_i| \, |\mu_j| \;\geq\; \left(\prod_{i \neq j} |\mu_i| \, |\mu_j| \right)^{1/[p(p-1)]} = \left(\prod_{i \neq j} |\mu_i|^{2(p-1)} \right)^{1/[p(p-1)]}$$

$$= \prod_{i \neq j} |\mu_i|^{2/p} = \Delta^{2/p} \, . \qquad \square \qquad (4)$$

Combining Equations (3) and (4) we arrive at the lower bound of Theorem 5.

By expanding $\sum\limits_{i=1}^{p} \sum\limits_{j=1}^{p} [|\mu_i| - |\mu_j|]^2$ and by taking into account (1), we obtain

$$p \sum_{i=1}^{p} \mu_i^2 - 2\, E_D(G)^2 + p \sum_{j=1}^{p} \mu_j^2$$

This expression is necessarily non-negative. The upper bound for E_D follows now from Lemma 7. $\square$

Theorem 6. Let G be an r-regular graph of diameter 2. Then

$$E_D \leq 2p - r - 2 + \sqrt{(p-1)\,[p\,(r+4) - (r+2)^2]} \, .$$

Proof. Let G be an r-regular graph with p vertices and q edges. Then by Theorem 4, the greatest D-eigenvalue is $\mu_1 = 2p - r - 2$. By applying the Cauchy–Schwarz inequality to the two $p-1$ vectors $(1, 1, \ldots, 1)$ and $(\mu_2, \mu_3, \ldots, \mu_p)$ we get

$$\left(\sum_{i=2}^{p} |\mu_i| \right)^2 \leq (p-1) \sum_{i=2}^{p} \mu_i^2$$

i. e.,

$$(E_D - \mu_1)^2 \leq (p-1) \left(4\,p^2 - 6q - 4p - \mu_1^2 \right)$$

i. e.,

$$E_D \leq \mu_1 + \sqrt{(p-1)\;(4\,p^2 - 6q - 4p - \mu_1^2)} \, .$$

Since $\mu_1 = 2p - r - 2$ and $2q = pr$, we have

$$E_D \leq 2p - r - 2 + \sqrt{(p-1)\;[p(r+4) - (r+2)^2]} \, . \qquad \square$$

Theorem 7. For any graph G of diameter 2,

$$E_D \leq \frac{1}{p}\left[2\,p^2 - 2q - 2p + \sqrt{(p-1)\,\left[(2p+q)\,(2\,p^2-4q) - 4\,p^2\right]}\right] .$$

Proof. This proof follows the ideas of Koolen and Moulton [22,23], used for obtaining an analogous upper bound for the ordinary graph Energy E. By the Cauchy–Schwarz inequality we have

$$E_D \leq \mu_1 + \sqrt{(p-1)\,[4\,p^2 - 6q - 4p - \mu_1^2]} .$$

Define a function

$$f(x) := x + \sqrt{(p-1)(4\,p^2 - 6q - 4p - x^2)}$$

for

$$\frac{2\,p^2 - 2q - 2p}{p} \leq x \leq \sqrt{4\,p^2 - 6q - 4p}$$

Then $(2\,p^2 - 2q - 2p)/p \geq 1$ and hence $f(x)$ is a decreasing function for $2\,p^2 - 2q - 2p/p \leq x^2$. But $(2\,p^2 - 2q - 2p)/p \leq x \leq x^2$ as $x \geq 1$. Hence $f(x) \leq f((2\,p^2 - 2q - 2p)/p)$, proving the theorem. $\square$

ON A PAIR OF D-EQUIENERGETIC GRAPHS

The problem of constructing non-cospectral graph having equal energies E, Eq. (2), has been much discussed and numerous examples of this kind were put forward [1,2,17–21,25–28]. Such pairs of graphs are referred to as "equienergetic" (the name first time used in [2]). Motivated by this, in this section we discuss the construction of D-equienergetic graphs. We succeed to do this for every $p \equiv 1\ (mod 3)$ and $p \equiv 0\ (mod\ 6)$.

Evidently, two graphs G_1 and G_2 are said to be D-equienergetic if $E_D(G_1) = E_D(G_2)$.

The graph $G \nabla G$ is obtained by joining every vertex of G to every vertex of another copy of G.

Theorem 8. Let G be a connected r-regular graph on p vertices with $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$. Then

$$spec_D(G \nabla G) = \begin{pmatrix} 3p - r - 2 & p - r - 2 & -2\,(\lambda_i + 2) \\ 1 & 1 & 2 \end{pmatrix}, \quad i = 2, \ldots, p\,.$$

Proof. The distance matrix of $G\nabla G$ can be written as

$$\begin{bmatrix} A + 2\overline{A} & J \\ J & A + 2\overline{A} \end{bmatrix}.$$

Then the theorem follows from Lemma 2. $\square$

Theorem 9. For every $p \equiv 0 \ (mod\ 6) \geq 18$, there exists a pair of D-equienergetic regular graphs.

Proof. Let $p = 6t$, $t \geq 3$. Let G_1 and G_2 be non-cospectral cubic graphs on $2t$ vertices as specified in Lemma 6. Then their line graphs $L(G_1)$ and $L(G_2)$ are 4-regular on $3t$ vertices. By Lemma 4, the only positive D-eigenvalues of $L(G_1)\nabla L(G_1)$ are $9t-6$ and $3t-6$. The same is true for $L(G_2)\nabla L(G_2)$. Thus $E_D(L(G_1)\nabla L(G_1)) = E_D(L(G_2)\nabla L(G_2)) = 24(t-1)$. The theorem follows now from the fact that both $L(G_1)\nabla L(G_1)$ and $L(G_2)\nabla L(G_2)$ have $6t$ vertices. $\square$

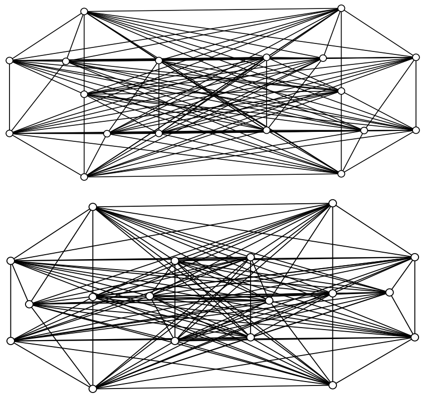

Figure 2: D-equienergetic graphs on 18 vertices with $E_D = 48$.

Theorem 10. For every $p \equiv 1 \ (mod\ 3) \geq 10$, there exists a pair of D-equienergetic graphs.

Proof. Let $p = 3t + 1$. Let G_1 and G_2 be non-cospectral cubic graphs on $2t$ vertices as specified by Lemma 6. The line graphs $L(G_1)$ and $L(G_2)$ possess $3t$ vertices and are regular of degree 4. Then by a similar argument as in Theorem 3, we have

$$spec_D(L(G)\nabla K_1) = \begin{pmatrix} 3t - 3 \pm \sqrt{9\,t^2 - 15t + 9} & -(\lambda_i + 2) \\ 1 & 1 \end{pmatrix}, \quad i = 2, \ldots, 3t$$

where λ_i , $i = 2, 3, \ldots, 3t$, are the (ordinary) eigenvalues of $L(G)$, different from its regularity. Since $\lambda_i + 2 \geq 0$ for $i = 2, \ldots, 3t$, and $3t - 3 \leq \sqrt{9\,t^2 - 15t + 9}$, we have

$$\begin{aligned} E_D(L(G)\nabla K_1) &= 2\sqrt{9\,t^2 - 15\,t + 9} + \sum_{i=2}^{3t}(\lambda_i + 2) \\ &= 2\sqrt{9\,t^2 - 15\,t + 9} - \lambda_1 + 2(3t - 1) \\ &= 2\sqrt{9\,t^2 - 15\,t + 9} - 4 + 2(3t - 1)\ . \end{aligned}$$

Thus

$$E_D(L(G_1)\nabla K_1) = E_D(L(G_2)\nabla K_2) = 2(3t - 3) + 2\sqrt{9\,t^2 - 15\,t + 9}$$

i. e., $L(G_1)\nabla K_1$ and $L(G_2)\nabla K_1$ are D-equienergetic. $\square$

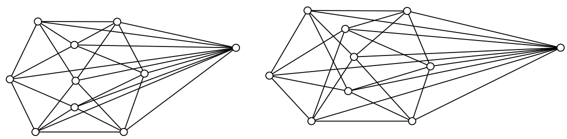

Figure 3: D-equienergetic graphs on 10 vertices with $E_D = 2(6 + 3\sqrt{5})$.

References

[1] R. Balakrishnan, The energy of a graph, *Lin. Algebra Appl.* **387** (2004) 287–295.

[2] V. Brankov, D. Stevanović, I. Gutman, Equienergetic chemical trees, *J. Serb. Chem. Soc.* **69** (2004) 549–553.

[3] F. Buckley, F. Harary, *Distance in Graphs*, Addison–Wesley, Redwood, 1990.

On the distance spectra of some graphs

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Abstract. *The D-eigenvalues of a connected graph G are the eigenvalues of its distance matrix D, and form the D-spectrum of G. The D-energy $E_D(G)$ of the graph G is the sum of the absolute values of its D-eigenvalues. Two (connected) graphs are said to be D-equienergetic if they have equal D-energies. The D-spectra of some graphs and their D-energies are calculated. A pair of D-equienergetic bipartite graphs on $24\,t$, $t \geq 3$, vertices is constructed.*

Key words: *distance eigenvalue (of a graph), distance spectrum (of a graph), distance energy (of a graph), distance–equienergetic graphs*

AMS subject classifications: 05C12, 05C50

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1. Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. The distance matrix $D = D(G)$ of G is defined so that its (i,j)-entry is equal to $d_G(v_i, v_j)$, the distance (= length of the shortest path [2]) between the vertices v_i and v_j of G. The eigenvalues of the $D(G)$ are said to be the D-eigenvalues of G and form the D-spectrum of G, denoted by $spec_D(G)$.

The ordinary graph spectrum is formed by the eigenvalues of the adjacency matrix [4]. In what follows we denote the ordinary eigenvalues of the graph G by λ_i , $i = 1, 2, \ldots, p$, and the respective spectrum by $spec(G)$.

Since the distance matrix is symmetric, all its eigenvalues μ_i , $i = 1, 2, \ldots, p$, are real and can be labelled so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. If $\mu_{i_1} > \mu_{i_2} > \cdots > \mu_{i_g}$ are the distinct D-eigenvalues, then the D-spectrum can be written as

$$spec_D(G) = \begin{pmatrix} \mu_{i_1} & \mu_{i_2} & \ldots & \mu_{i_g} \\ m_1 & m_2 & \ldots & m_g \end{pmatrix}$$

where m_j indicates the algebraic multiplicity of the eigenvalue μ_{i_j}. Of course, $m_1 + m_2 + \cdots + m_g = p$.

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Two graphs G and H for which $spec_D(G) = spec_D(H)$ are said to be D-cospectral. Otherwise, they are non-D-cospectral.

The D-energy, $E_D(G)$, of G is defined as

$$E_D(G) = \sum_{i=1}^{p} |\mu_i| \ . \tag{1}$$

Two graphs with equal D-energy are said to be D-equienergetic. D-cospectral graphs are evidently D-equienergetic. Therefore, in what follows we focus our attention to D-equienergetic non-D-cospectral graphs.

The concept of D-energy, Eq. (1), was recently introduced [11]. This definition was motivated by the much older [7] and nowadays extensively studied [8, 9, 10, 13, 14, 15, 16] graph energy, defined in a manner fully analogous to Eq. (1), but in terms of the ordinary graph eigenvalues (eigenvalues of the adjacency matrix, see [4]).

In this paper we first derive a Hoffman–type relation for the distance matrix of distance regular graphs. By means of it, the distance spectra of some graphs and their energies are obtained. Also pairs of D-equienergetic bipartite graphs on $24\,t$, $t \geq 3$, vertices are constructed. All graphs considered in this paper are simple and we follow [4] for spectral graph theoretic terminology.

The considerations in the subsequent sections are based on the applications of the following lemmas:

Lemma 1 [see [4]]. *Let G be a graph with adjacency matrix A and $spec(G) = \{\lambda_1, \lambda_2, \dots, \lambda_p\}$. Then* $\det A = \prod_{i=1}^{p} \lambda_i$. *In addition, for any polynomial $P(x)$, $P(\lambda)$ is an eigenvalue of $P(A)$ and hence* $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

Lemma 2 [see [5]]. *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a 2×2 block symmetric matrix. Then the eigenvalues of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

Lemma 3 [see [4]]. *Let M, N, P, and Q be matrices, and let M be invertible. Let*

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix} .$$

Then $\det S = \det M \det(Q - PM^{-1}N)$. *Besides, if M and P commute, then* $\det S = \det(MQ - PN)$.

Lemma 4 [see [4]]. *Let G be a connected r-regular graph, $r \geq 3$, with ordinary spectrum $spec(G)\{r, \lambda_2, \dots, \lambda_p\}$. Then*

$$spec(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \cdots & \lambda_p + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & p(r-2)/2 \end{pmatrix} .$$

Lemma 5 [see [4]]. *For every $t \geq 3$, there exists a pair of non-cospectral cubic graphs on $2t$ vertices.*

Lemma 6 [see [6]]. *The distance spectrum of the cycle C_n is given by*

n	*greatest eigenvalue*	*j even*	*j odd*
even	$\frac{n^2}{4}$	*0*	$-cosec^2\left(\frac{\pi j}{n}\right)$
odd	$\frac{n^2-1}{4}$	$-\frac{1}{4}sec^2\left(\frac{\pi j}{2n}\right)$	$-\frac{1}{4}cosec^2\left(\frac{\pi j}{2n}\right)$

Definition 1 [see [12]]. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_p\}$. Take another copy of G with the vertices labelled by $\{u_1, u_2, \dots, u_p\}$ where u_i corresponds to v_i for each i. Make u_i adjacent to all the vertices in $N(v_i)$ in G, for each i. The resulting graph, denoted by D_2G, is called the double graph of G.*

Definition 2 [see [4]]. *Let G be a graph. Attach a pendant vertex to each vertex of G. The resulting graph, denoted by $G \circ K_1$, is called the corona of G with K_1.*

We first prove the following auxiliary theorem.

Theorem 1. *Let M be a real symmetric irreducible square matrix of order p in which each row sum is equal to a constant k.Then there exists a polynomial $Q(x)$ such that $Q(M) = J$, where J is the all one square matrix whose order is same as that of M.*

Proof. Since M is a real symmetric irreducible matrix in which each row sums to k, by the Frobenius theorem [4], k is a simple and greatest eigenvalue of M. The matrix M is diagonalizable because it is real and symmetric. Therefore there exists an orthonormal basis of characteristic vectors of M, associated with the eigenvalues of M.

Let $\lambda_1 = k, \lambda_2, \dots, \lambda_g$ be the distinct eigenvalues of M. Let $\Im(\lambda_i)$ be the eigenspace spanned by the orthonormal set of characteristic vectors $\{x_1^i, x_2^i, \dots, x_{p_i}^i\}$ associated with λ_i, $i = 1, 2, \dots, g$. Then M has a spectral decomposition

$$M = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_g T_g$$

where T_i is the projection of M onto $\Im(\lambda_i)$, treating M as a linear operator. Then $T_i^2 = T_i$, $T_i T_j = 0$, $i \neq j$ and

$$T_i = x_1^i \left(x_1^i\right)^T + x_2^i \left(x_2^i\right)^T + \cdots + x_{p_i}^i \left(x_{p_i}^i\right)^T .$$

Now, corresponding to the greatest eigenvalue k of M, there exists a unique (one-dimensional) orthonormal basis

$$x_1 = \begin{bmatrix} 1/\sqrt{p} \\ 1/\sqrt{p} \\ \vdots \\ 1/\sqrt{p} \end{bmatrix}$$

for $\Im(\lambda_1) = \Im(k)$, such that $M = k\,T_1 + \lambda_2\,T_2 + \cdots + \lambda_g\,T_g$ where

$$\begin{aligned} T_1 &= \begin{bmatrix} 1/\sqrt{p} \\ 1/\sqrt{p} \\ \vdots \\ 1/\sqrt{p} \end{bmatrix} \begin{bmatrix} 1/\sqrt{p}, & 1/\sqrt{p}, & \cdots, & 1/\sqrt{p} \end{bmatrix} \\ &= \begin{bmatrix} 1/p & 1/p & \cdots & 1/p \\ 1/p & 1/p & \cdots & 1/p \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1/p & 1/p & \cdots & 1/p \end{bmatrix} = \frac{1}{p}\,J\,. \end{aligned}$$

Because the T_i's are projections, we have $f(M) = f(k)\,T_1 + f(\lambda_2)\,T_2 + \cdots + f(\lambda_g)\,T_g$ for any polynomial $f(x)$. As M is diagonalizable, the minimal polynomial of M is $(x-k)(x-\lambda_2)\cdots(x-\lambda_g)$.

Let $S(x) = (x-\lambda_2)\cdots(x-\lambda_g)$. Then $S(\lambda_i) = 0$, $\lambda_i \neq k$. Thus $S(M) = S(k)\,T_1 S(k)\,(1/p)\,J$. Choose $Q(x) = p\,S(x)/S(k)$. This $Q(x)$ satisfies the requirement of the theorem. □

Theorem 2. *Let D be the distance matrix of a connected distance regular graph G. Then D is irreducible and there exists a polynomial $P(x)$ such that $P(D) = J$. In this case*

$$P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\cdots(x-\lambda_g)}{(k-\lambda_2)(k-\lambda_3)\ldots(k-\lambda_g)}$$

where k is the unique sum of each row which is also the greatest simple eigenvalue of D, whereas $\lambda_2, \lambda_3, \ldots, \lambda_g$ are the other distinct eigenvalues of D.

Proof. The theorem follows from Theorem 1 due to the observation that the distance matrix of a connected distance regular graph is irreducible, symmetric and each row sums to a constant. □

The rest of this paper is organized as follows. In the next section we obtain the distance spectra of $D_2(G)$, $G \times K_2$, $G[K_2]$, the lexicographic product of G with K_2, and $G \circ K_1$. Using this, the distance energies of $D_2(C_{2n})$, $C_n \times K_2$, $C_{2n}[K_2]$, and $C_n \circ K_1$ are calculated. In the third section the D-spectrum of the extended double cover graphs of regular graphs of diameter 2 is discussed and a pair of D-equienergetic bipartite graphs on $24t$, $t \geq 3$ vertices is constructed.

For operations on graphs that are not defined in this paper see [4].

2. Distance spectra of some graphs

In this section we obtain the distance spectra of the double graph of C_n, the Cartesian product of C_n with K_2 and the corona of C_n with K_1.

2.1. The double graph of G

Theorem 3. *Let G be a graph with distance spectrum $spec_D(G) = \{\mu_1, \mu_2, \dots, \mu_p\}$. Then*

$$spec_D(D_2G) = \begin{pmatrix} 2\,(\mu_i + 1) & -2 \\ 1 & p \end{pmatrix}\ ,\ i = 1, 2, \dots, p\ .$$

Proof. By definition of $D_2(G)$ we have:

$$\begin{aligned} d_{D_2G}(v_i, v_j) &= d_G(v_i, v_j) \\ d_{D_2G}(v_i, u_i) &= 2 \\ d_{D_2G}(v_i, u_j) &= d_G(v_i, v_j) \\ d_{D_2G}(v_j, u_i) &= d_G(v_j, v_i)\ . \end{aligned}$$

Hence a suitable ordering of vertices yields the distance matrix of D_2G of the form

$$\begin{bmatrix} D & D + 2I \\ D + 2I & D \end{bmatrix}$$

and the theorem follows from Lemma 2. □

Theorem 4. $E_D(D_2C_{2n}) = 4n(n+1)$.
Proof. By Lemma 6 and Theorem 3 we have

$$spec_D(D_2C_{2n}) = \begin{pmatrix} 2\left(n^2 + 1\right) & 2 & -2\cot^2(\pi j/2n) & -2 \\ 1 & n-1 & 1 & 2n \end{pmatrix}\ ,\ j = 1, 3, 5, \dots, 2n{-}1\ .$$

Thus $E_D(D_2C_{2n}) = 2 \times [2(n^2+1) + 2(n-1)]4n(n+1)$. □

2.2. The Cartesian product $G \times K_2$

Theorem 5. *Let G be a distance regular graph with distance regularity k, distance matrix D, and D-spectrum $\{\mu_1 = k, \mu_2, \dots, \mu_p\}$. Then*

$$spec_D(G \times K_2) = \begin{pmatrix} 2k + p & -p & 2\,\mu_i & 0 \\ 1 & 1 & 1 & p-1 \end{pmatrix}\ ,\ i = 2, 3, \dots, p\ .$$

Proof. The theorem follows from the fact that the distance matrix of $G \times K_2$ has the form

$$\begin{bmatrix} D & D + J \\ D + J & D \end{bmatrix}$$

and from Theorem 1 and Lemma 2. □

Corollary 1. $E_D(G \times K_2) = 2(E_D(G) + p)$.

2.3. The corona of G and K_1

Theorem 6. *Let G be a connected distance regular graph with distance regularity k, distance matrix D, and $spec_D(G) = \{\mu_1 = k, \mu_2, \ldots, \mu_p\}$. Then $spec_D(G \circ K_1)$ consists of the numbers*

$$p+k-1+\sqrt{(p+k)^2+(p-1)^2} \qquad , \qquad p+k-1-\sqrt{(p+k)^2+(p-1)^2}$$

$$\mu_i - 1 + \sqrt{\mu_i^2+1} \;\; , \;\; \mu_i - 1 - \sqrt{\mu_i^2+1} \qquad , \qquad i = 2,3,\ldots,p \, .$$

Proof. From the definition of $G \circ K_1$, it follows that the distance matrix H of $G \circ K_1$ is of the form

$$\begin{bmatrix} D & D+J \\ D+J & D+2(J-I) \end{bmatrix} .$$

Now the characteristic equation of H is

$$\begin{aligned} |\lambda I - H| &= 0 \Rightarrow \begin{vmatrix} \lambda I - D & -(D+J) \\ -(D+J) & \lambda I - D - 2(J-I) \end{vmatrix} = 0 \\ &\Rightarrow \left| (\lambda I - D)(\lambda I - D - 2(J-I)) - (D+J)^2 \right| = 0 \text{ by Lemma 3} \end{aligned}$$

Now D being the distance matrix of a distance regular graph, it satisfies the requirement in Theorem 2. Then the $D-$ spectrum of $G \circ K_1$ follows from Theorem 2 and Lemma 1. □

Corollary 2.

$$\begin{aligned} E_D(C_{2n} \circ K_1) &= 2\left[(n-1)^2 + \sqrt{(n-1)^4 + 6\,n^2}\right] \\ E_D(C_{2n+1} \circ K_1) &= 2\left[n^2 + 3n + \sqrt{(n^2+3n)^2 + 6\,n^2 + 6n + 1}\right] . \end{aligned}$$

2.4. The lexicographic product of G with K_2

Theorem 7. *Let G be a connected graph with distance spectrum $spec_D(G)\{\mu_1 = k, \mu_2, \ldots, \mu_p\}$. Then*

$$spec_D(G[K_2]) = \begin{pmatrix} 2\mu_i + 1 & -1 \\ 1 & p \end{pmatrix} , \; i = 1,2,\ldots,p \, .$$

Proof. From the definition of the lexicographic product of G with K_2, its distance matrix can be written as

$$\begin{bmatrix} D & D+I \\ D+I & D \end{bmatrix}$$

and the theorem follows from Lemma 2. □

Corollary 3. $E_D(C_{2n}[K_2]) = 2n(2n+1)$.

Proof. From Lemma 6 and Theorem 7 we have

$$spec_D(C_{2n}[K_2]) = \begin{pmatrix} 2\,n^2+1 & 1 & -1 & 1-2\,cosec^2(\pi j/2n) \\ 1 & n-1 & 2n & 1 \end{pmatrix} \,,\; j=1,3,5,\dots\,.$$

Since $1-2\,cosec^2\theta = -(cot^2\theta + coesc^2\theta)$, the only positive eigenvalues are $2\,n^2+1$ and 1 with multiplicities 1 and $n-1$, respectively. Thus $E_D(C_{2n}[K_2]) = 2n(2n+1)$. □

3. The extended double cover graph of regular graphs of diameter 2

In [1] N. Alon introduced the concept of extended double cover graph of a graph as follows.

Let G be a graph on the vertex set $\{v_1, v_2, \dots, v_p\}$. Define a bipartite graph H with $V(H) = \{v_1, v_2, \dots, v_p, u_1, u_2, \dots, u_p\}$ in which v_i is adjacent to u_i for each $i = 1, 2, \dots, p$ and v_i is adjacent to u_j if v_i is adjacent to v_j in G . The graph H is known as the extended double cover graph (EDC-graph) of G . The ordinary spectrum of H has been determined in [3].

In this section we obtain the distance spectrum of the $EDC-$graph of a regular graph of diameter 2 and use it to construct regular D-equienergetic bipartite graphs on $24\,t$ vertices, for $t \geq 3$.

Theorem 8. *Let G be an r-regular graph of diameter 2 on p vertices with (ordinary) spectrum $\{r, \lambda_2, \dots, \lambda_p\}$. Then the D-spectrum of the EDC-graph of G consists of the numbers $5p-2r-4$, $2r-p$, $-2\,(\lambda_i+2)$, $i = 2, 3, \dots, p$, and $2\,\lambda_i$, $i = 2, 3, \dots, p$.*

Proof. Let A and $\overline{A}$ be, respectively, the adjacency matrices of G and $\overline{G}$. Then by the definition of the EDC-graph, its distance matrix can be written as

$$\begin{bmatrix} 2\,(J-I) & A+3\overline{A}+I \\ A+3\overline{A}+I & 2\,(J-I) \end{bmatrix}$$

and the theorem follows from Lemmas 1 and 3 and also from the observation that $\overline{A} = J-I-A$. □

Corollary 4.

$$E_D\,(EDC\,(C_p \nabla C_p)) = \begin{cases} 40 \;, p=3 \\ 4\,[E\,(C_p)+5p-10] \;, p \geqslant 4 \end{cases}$$

where $C_p \nabla C_p$ is the join [4] of C_p with itself.

Proof. The join of C_p with itself is a regular graph diameter 2 with the ordinary spectrum

$$\begin{pmatrix} p+2 & 2-p & \lambda_i \\ 1 & 1 & 2 \end{pmatrix} \,,\; i = 2, 3, \dots, p$$

where $\{2, \lambda_2, \dots, \lambda_p\}$ is the ordinary spectrum of C_p. Then by the above theorem, the distance spectrum of $EDC(C_p \nabla C_p)$ is

$$\begin{pmatrix} 8p-8 & 4 & -2(\lambda_i+2) & 2p-8 & 4-2p & 2\lambda_i \\ 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}, \quad i = 2, 3, \dots, p$$

and hence the corollary follows as $E(C_3) = 4$. □

3.1. On a pair of D-equienergetic bipartite graphs

Theorem 9. *There exists a pair of regular non-D-cospectral D-equienergetic bipartite graphs on $24t$ vertices, for each $t \geq 3$.*

Proof. Let G be a cubic graph on $2t$ vertices, $t \geq 3$. Consider $L^2(G)$, its second iterated line graph. Then by Lemma 4 and Theorem 8, we calculate that for $F = L^2(G) \nabla L^2(G)$, the D-spectrum of $EDC(F)$ is

$$\begin{pmatrix} 16(3t-1) & 12 & 0 & 2(\lambda_i+3) & 12t-16 & -4 & -12(t-1) & -2(\lambda_i+5) \\ 1 & 1 & 8t & 2 & 1 & 8t & 1 & 2 \end{pmatrix},$$

$i = 2, 3, \dots, 2t$. Thus

$$\begin{aligned} E_D(EDC(F)) &= 2 \times \left[12(t-1) + 32\,t + 4 \sum_{i=2}^{2t} (\lambda_i + 5) \right] \\ &= 2 \times [12\,t - 12 + 32\,t + 4(-3 + 5(2t-1))] \\ &= 8\,(21\,t - 11)\,. \end{aligned}$$

Now let G_1 and G_2 be the two non-cospectral cubic graphs on $2t$ vertices as given by Lemma 5. Further, let H_1 and H_2 be the EDC-graphs of $L^2(G_1) \nabla L^2(G_1)$ and $L^2(G_2) \nabla L^2(G_2)$, respectively. Then H_1 and H_2 are bipartite and $E_D(H_1) = E_D(H_2) = 8\,(21\,t - 11)$, proving the theorem. □

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References

[1] N. ALON, *Eigenvalues and expanders*, Combinatorica **6**(1986), 83–96.

[2] F. BUCKLEY, F. HARARY, *Distance in Graphs*, Addison–Wesley, Redwood, 1990.

[3] Z. CHEN, *Spectra of extended double cover graphs*, Czechoslovak Math. J. **54**(2004), 1077–1082.

[4] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Applications*, Academic Press, New York, 1980.

[5] P. J. Davis, *Circulant Matrices*, Wiley, New York, 1979.

[6] P. W. Fowler, G. Caporossi, P. Hansen, *Distance matrices, Wiener indices, and related invariants of fullerenes*, J. Phys. Chem. A **105**(2001), 6232–6242.

[7] I. Gutman, *The energy of a graph: Old and new results*, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann(Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, 2001, pp. 196–211.

[8] I. Gutman, *On graphs whose energy exceeds the number of vertices*, Lin. Algebra Appl., in press.

[9] I. Gutman, S. Zare Firoozabadi, J. A. de la Peña, J. Rada, *On the energy of regular graphs*, MATCH Commun. Math. Comput. Chem. **57**(2007), 435–442.

[10] W. H. Haemers, *Strongly regular graphs with maximal energy*, Lin. Algebra Appl., in press.

[11] G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **60**(2008), in press.

[12] G. Indulal, A. Vijayakumar, *On a pair of equienergetic graphs*, MATCH Commun. Math. Comput. Chem. **55**(2006), 83–90.

[13] G. Indulal, A. Vijayakumar, *A note on energy of some graphs*, MATCH Commun. Math. Comput. Chem. **59**(2008), 269–274.

[14] X. Li, J. Zhang, *On bicyclic graphs with maximal energy*, Lin. Algebra Appl. **427**(2007), 87–98.

[15] V. Nikiforov, *The energy of graphs and matrices*, J. Math. Anal. Appl. **326**(2007), 1472–1475.

[16] I. Shparlinski, *On the energy of some circulant graphs*, Lin. Algebra Appl. **414**(2006), 378–382.

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Distance spectrum of graph compositions*

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Abstract

The D-eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_p\}$ of a graph G are the eigenvalues of its distance matrix D and form the distance spectrum or D-spectrum of G denoted by $spec_D(G)$. In this paper we obtain the D-spectrum of the cartesian product of two distance regular graphs. The D-spectrum of the lexicographic product $G[H]$ of two graphs G and H when H is regular is also obtained. The D-eigenvalues of the Hamming graphs $Ham(d, n)$ of diameter d and order n^d and those of the C_4 nanotori, T_{k,m,C_4} are determined.

Keywords: Distance spectrum, Cartesian product, lexicographic product, Hamming graphs, C_4 nanotori.

Math. Subj. Class.: 05C12, 05C50

1 Introduction

Adjacency matrix of a graph and its spectrum have arisen as a natural tool with which one can study graphs and its structural properties. Also the adjacency spectrum find applications in quantum theory and chemistry [3]. The idea of distance matrix seems a natural generalization, with perhaps more specificity than that of an adjacency matrix. Distance matrix and their spectra have arisen independently from a data communication problem [7] studied by Graham and Pollack in 1971 in which the most important feature is the number of negative eigenvalues of the distance matrix. While the problem of computing the characteristic polynomial of adjacency matrix and its spectrum appears to be solved for many large graphs, the related distance polynomials have received much less attention. The distance matrix is more complex than the ordinary adjacency matrix of a graph since the distance matrix is a complete matrix (dense) while the adjacency matrix often is very sparse. Thus the computation of the characteristic polynomial of the distance matrix is

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computationally a much more intense problem and, in general, there are no simple analytical solutions except for a few trees [6]. For this reason, distance polynomials of only trees have been studied extensively in the mathematical literature [6, 16].

The distance matrix of a graph has numerous applications to chemistry and other branches of science. The distance matrix, contains information on various walks and self-avoiding walks of chemical graphs, is immensely useful in the computation of topological indices such as the Wiener index, is useful in the computation of thermodynamic properties such as pressure and temperature coefficients and it contains more structural information compared to a simple adjacency matrix. In addition to such applications in chemical sciences, distance matrices find applications in music theory, ornithology, molecular biology, psychology, archeology etc. For a survey see [1] and also the papers cited therein.

Let G be a connected graph with vertex set $V(G) = \{u_1, u_2, \ldots, \ldots, u_p\}$. The distance matrix $D = D(G)$ of G is defined so that its (i, j)-entry is equal to $d_G(u_i, u_j)$, the distance (= length of the shortest path [2]) between the vertices u_i and u_j of G. The eigenvalues of $D(G)$ are said to be the D-eigenvalues of G and form the distance spectrum or the D-spectrum of G , denoted by $spec_D(G)$.

The characteristic polynomial of the D-matrix and the corresponding spectra have been considered in [4, 6, 7, 8]. For some recent works on D-spectrum see [9, 10, 11, 12, 13, 18].

For two graphs, the ordinary spectrum of graph compositions is well explored and generalized results of NEPS of graphs are presented in [3]. Such studies for the distance spectrum did not appear in literature yet and hence in this paper we present the following.

Let G and H be two graphs. Let $G + H$ and $G[H]$ denote the cartesian product and lexicographic product of G and H respectively [3].

In this paper we first derive the D-spectrum of $G + H$ and $G[H]$. By means of this, the distance spectrum of the Hamming graph and C_4 nanotori are obtained. A work of this type is reported here for the first time.

All graphs considered in this paper are simple and we follow [3] for spectral graph theoretic terminology and [2] for distance in graphs. The considerations in the subsequent sections are based on the applications of the following lemmas.

Lemma 1.1 ([3]). *Let G be an r-regular graph on p vertices with adjacency eigenvalues $r, \lambda_2, \ldots, \lambda_p$. Then G and its complement $\overline{G}$ have the same eigenvectors, and the eigenvalues of $\overline{G}$ are $p - r - 1, -1 - \lambda_2, \ldots, -1 - \lambda_p$.*

Lemma 1.2 ([5]). *The distance spectrum of the cycle C_n is given by*

n	*greatest eigenvalue*	*j even*	*j odd*
even	$\frac{n^2}{4}$	*0*	$-\operatorname{cosec}^2\left(\frac{\pi j}{n}\right)$
odd	$\frac{n^2-1}{4}$	$-\frac{1}{4}\sec^2\left(\frac{\pi j}{2n}\right)$	$-\frac{1}{4}\operatorname{cosec}^2\left(\frac{\pi j}{2n}\right)$

Definition 1.3 ([14]). The Hamming graph $Ham(d,n), d \geq 2, n \geq 2$, of diameter d and characteristic n have vertex set consisting of all d-tuples of elements taken from an n-element set, with two vertices adjacent if and only if they differ in exactly one coordinate. $Ham(d,n)$ is equal to $\underbrace{K_n + K_n + \cdots + K_n}_{d}$, the cartesian product of K_n, the complete graph on n vertices, d times. $Ham(3,n)$ is referred to as a cubic lattice graph.

Lemma 1.4 ([17]). *Let G and H be two connected graphs, and let $u = (u_1, u_2),\ v = (v_1, v_2) \in V(G) \times V(H)$. Let $G + H$ denote their cartesian product. Then*

$$d_{G+H}(u,v) = d_G(u_1, v_1) + d_H(u_2, v_2).$$

2 The D-spectrum of $G + H$

In this section we derive the D-spectrum of the cartesian product of two distance regular graphs.

Theorem 2.1. *Let G and H be two distance regular graphs on p and n vertices with distance regularity k and t respectively. Let $spec_D(G) = \{k, \mu_2, \mu_3, \ldots, \mu_p\}$ and $spec_D(H) = \{t, \eta_2, \eta_3, \ldots, \eta_n\}$. Then*

$$spec_D(G + H) = \{nk + pt, n\mu_i, p\eta_j, 0\}$$

$i = 2, \ldots, p\ ,\ j = 2, \ldots, n$ *and* 0 *is with multiplicity* $(p-1)(n-1)$.

Proof. Let D_G and D_H be the distance matrices of G and H respectively. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$. Then $D_G = [d_{ij}]$ and $D_H = [e_{ij}]$ where $d_{ij} = d_G(u_i, u_j)$ and $e_{ij} = d_H(v_i, v_j)$. Since G and H are distance regular graphs with distance regularities k and t respectively, we have

$$\sum_{j=1}^{p} d_{rj} = k \quad \text{and} \quad \sum_{j=1}^{n} e_{qj} = t \tag{2.1}$$

Also since G is distance regular, the all one column vector of order $p \times 1$ is the eigenvector corresponding to the greatest eigenvalue k of D_G. As D_G is real and symmetric, it is diagonalizable and hence admits an orthogonal basis B_G consisting of eigenvectors corresponding to its eigenvalues. Thus if μ_i is an eigenvalue of D_G which is different from k with an eigenvector $X_i = \left[x_i^1, x_i^2, \ldots, x_i^p\right]^T \in B_G$, then $\sum\limits_{j=1}^{p} x_i^j = 0$.

Let $u = (u_1, u_2),\ v = (v_1, v_2) \in V(G) \times V(H)$. Then by Lemma 1.4

$$d_{G+H}(u,v) = d_G(u_1, v_1) + d_H(u_2, v_2).$$

By a suitable ordering of vertices in $G + H$ and by virtue of Lemma 1.4, its D-matrix, C can be written in the form

$$C=\begin{bmatrix} d_{11}+e_{11} & \cdots & d_{11}+e_{1n} & \cdots & \cdots & d_{1p}+e_{11} & \cdots & d_{1p}+e_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{11}+e_{n1} & \cdots & d_{11}+e_{nn} & \cdots & \cdots & d_{1p}+e_{n1} & \cdots & d_{1p}+e_{nn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{p1}+e_{11} & \cdots & d_{p1}+e_{1n} & \cdots & \cdots & d_{pp}+e_{11} & \cdots & d_{pp}+e_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{p1}+e_{1n} & \vdots & d_{p1}+e_{nn} & \cdots & \cdots & d_{pp}+e_{n1} & \cdots & d_{pp}+e_{nn} \end{bmatrix}$$

$$=\begin{bmatrix} d(u_1,u_1)\cdot J_n+D_H & d(u_1,u_2)\cdot J_n+D_H & \cdots & \cdots & d(u_1,u_p)\cdot J_n+D_H \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d(u_p,u_1)\cdot J_n+D_H & d(u_p,u_2)\cdot J_n+D_H & \cdots & \cdots & d(u_p,u_p)\cdot J_n+D_H \end{bmatrix}$$

$$= D_G \otimes J_n + J_p \otimes D_H$$

where $\otimes$ denotes the tensor product of matrices.
Now we find the eigenvalues of C by considering eigenvectors associated with them. The following relation for matrices is well known [15]. For the matrices A, B, C and D

$$(A \otimes B)\cdot(C \otimes D) = (AC) \otimes (BD)$$

whenever the products AC and BD exist.

Let $\mathbf{1}_G$ denote the all one eigenvector corresponding to the eigenvalue k of G and $\mathbf{1}_H$ the all one eigenvector corresponding to the eigenvalue t of H. Then

$$D_G \cdot \mathbf{1}_G = k\mathbf{1}_G \quad \text{and} \quad D_H \cdot \mathbf{1}_H = t\mathbf{1}_H$$

Therefore

$$\begin{aligned} C\cdot(\mathbf{1}_G \otimes \mathbf{1}_H) &= (D_G \otimes J_n + J_p \otimes D_H)\cdot(\mathbf{1}_G \otimes \mathbf{1}_H) \\ &= (D_G \cdot \mathbf{1}_G) \otimes (J_n\mathbf{1}_H) + (J_p\mathbf{1}_G) \otimes (D_H \cdot \mathbf{1}_H) \\ &= k\mathbf{1}_G \otimes n\mathbf{1}_H + p\mathbf{1}_G \otimes t\mathbf{1}_H \\ &= nk\cdot(\mathbf{1}_G \otimes \mathbf{1}_H) + pt\cdot(\mathbf{1}_G \otimes \mathbf{1}_H) \\ &= (nk+pt)\cdot(\mathbf{1}_G \otimes \mathbf{1}_H) \end{aligned}$$

showing that $\mathbf{1}_G \otimes \mathbf{1}_H$ is the eigenvector corresponding to the eigenvalue $nk + pt$ of C.

Let X_i be the eigenvector corresponding to the eigenvalue μ_i of D_G. Then $X_i \otimes \mathbf{1}_H$ is the eigenvector corresponding to the eigenvalue $n\mu_i$ of C. For

$$\begin{aligned} C\cdot(X_i \otimes \mathbf{1}_H) &= (D_G \otimes J_n + J_p \otimes D_H)\cdot(X_i \otimes \mathbf{1}_H) \\ &= (D_G \cdot X_i) \otimes (J_n\mathbf{1}_H) + (J_pX_i) \otimes (D_H \cdot \mathbf{1}_H) \\ &= \mu_iX_i \otimes n\mathbf{1}_H + 0 \otimes t\mathbf{1}_H \\ &= n\mu_i(X_i \otimes \mathbf{1}_H) \end{aligned}$$

Similarly if Z_j is an eigenvector corresponding to the eigenvalue η_j of D_H, then $\mathbf{1}_G \otimes Z_j$ is an eigenvector corresponding to the eigenvalue $p\eta_j$ of C.

In addition to these eigenvalues we can see that 0 appears to be an eigenvalue with multiplicity $(p-1)(n-1)$. For let $R_p^i,\ i = 2, 3, \ldots, p$ be the $(p-1)$ linearly independent eigenvectors corresponding to the eigenvalue 0 of J_p and $T_n^j,\ j = 2, 3, \ldots, n-1$ be the $(n-1)$ linearly independent eigenvectors corresponding to the eigenvalue 0 of J_n. Then the $(p-1)(n-1)$ vectors $R_p^i \otimes T_n^j$ are linearly independent and are the eigenvectors corresponding to 0 of C. For

$$\begin{aligned}
C \cdot \left(R_p^i \otimes T_n^j\right) &= (D_G \otimes J_n + J_p \otimes D_H) \cdot \left(R_p^i \otimes T_n^j\right) \\
&= \left(D_G \cdot R_p^i\right) \otimes \left(J_n \cdot T_n^j\right) + \left(J_p R_p^i\right) \otimes \left(D_H \cdot T_n^j\right) \\
&= \left(D_G \cdot R_p^i\right) \otimes 0 + 0 \otimes \left(D_H \cdot T_n^j\right) \\
&= 0
\end{aligned}$$

Now the pn vectors $X_i \otimes \mathbf{1}_H, \mathbf{1}_G \otimes Z_j$ and $R_p^i \otimes T_n^j$ are linearly independent and as C has a basis consisting of linearly independent eigenvectors, the theorem follows. □

2.1 The D-spectrum of $Ham(d, n)$

In [14], the ordinary spectrum of the cubic lattice graph is obtained. In this section we use Theorem 2.1 to obtain the D-spectrum of $Ham(d, n)$.

Theorem 2.2. *Let $Ham(d, n)$ be the Hamming graph of characteristic n. Then the D-eigenvalues of $Ham(d, n)$ are $dn^{d-1}(n-1)$, 0 and $-n^{d-1}$ with multiplicities 1, $n^d - d(n-1) - 1$ and $d\,(n-1)$ respectively.*

Proof. The graph K_n is distance regular with distance regularity $n-1$. Now the proof follows by repeated application of Theorem 2.1 and from the ordinary spectrum of K_n [3]. □

2.2 The D-spectrum of the C_4 nanotori, T_{k,m,C_4}

The graph $C_k + C_m$ where both k and m are odd is defined as the C_4 nanotori, T_{k,m,C_4}.

Theorem 2.3. *The distance spectrum of the C_4 nanotori, T_{k,m,C_4} consists of the following numbers*

$$\frac{(m+k)(mk-1)}{4},\ -\frac{m}{4}\sec^2\left(\frac{\pi j}{2k}\right),\ -\frac{m}{4}\operatorname{cosec}^2\left(\frac{\pi r}{2k}\right),$$
$$-\frac{k}{4}\sec^2\left(\frac{\pi t}{2m}\right),\ -\frac{k}{4}\operatorname{cosec}^2\left(\frac{\pi l}{2m}\right)$$

where $j \in \{1, 2, \ldots, k-1\}$ and even, $r \in \{1, 2, \ldots, k-1\}$ and odd $t \in \{1, 2, \ldots, m-1\}$ and even and $l \in \{1, 2, \ldots, m-1\}$ and odd together with 0 of multiplicity $(m-1)(k-1)$.

Proof. The cycle C_{2n+1} is distance regular with distance regularity $n(n+1)$. Now the proof follows from Theorem 2.1 and Lemma 1.2. □

3 The D-spectrum of $G[H]$

In this section we obtain the distance spectrum of the lexicographic product $G[H]$ of two graphs G and H. The following definition of the lexicographic product of G and H is from [3].

Definition 3.1. Let G and H be two graphs on vertex sets $V(G) = \{u_1, u_2, \ldots, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, \ldots, v_n\}$ respectively. Then their lexicographic product $G[H]$ is a graph defined by $V(G[H]) = V(G) \times V(H)$, the cartesian product of $V(G)$ and $V(H)$ in which $u = (u_1, v_1)$ be adjacent to $v = (u_2, v_2)$ if and only if either

1. u_1 be adjacent to v_1 in G or
2. $u_1 = v_1$ and u_2 be adjacent to v_2 in G.

Distance in $G[H]$

We prove the following lemma on distance in lexicographic product of graphs.

Lemma 3.2. *Let G and H be two connected graphs with atleast two vertices and let* $u = (u_1, v_1)$, $v = (u_2, v_2) \in V(G) \times V(H)$. *Then*

$$d_{G[H]}(u, v) = \begin{cases} d_G(u_1, u_2) \text{ if } u_1 \neq u_2 \\ 1 \text{ if } u_1 = u_2 \text{ and } v_1 \text{ adjacent to } v_2 \\ 2 \text{ if } u_1 = u_2 \text{ and } v_1 \text{ not adjacent to } v_2 \end{cases}$$

Proof. We show that in the corresponding composition there exist a path between u and v of length as given in the lemma. Let $d_G(u_1, u_2) = t$ and $u_1 = s_0, s_1, \ldots, s_t = u_2$ be the shortest $u_1 - u_2$ path in G.
Let $u = (u_1, v_1)$, $v = (u_2, v_2) \in V(G) \times V(H)$ and $u_1 \neq u_2$. Since the successive ordered pairs in any $u - v$ path can change both the coordinates and also as u_2 is reachable from u_1 by not less that t steps, any $u - v$ path in $G[H]$ is of length atleast t.

Now the following $u - v$ path in $G[H]$ is of length t.

$P : u = (s_0, v_1), (s_1, v_2), (s_2, v_2), \ldots, (s_t, v_2) = v$. Thus $d_{G[H]}(u, v) = d_G(u_1, u_2)$ if $u_1 \neq u_2$.

Now suppose $u_1 = u_2$ and v_1 be adjacent to v_2. Then by the definition of $G[H]$, we have $d_{G[H]}(u, v) = 1$.

Now suppose $u_1 = u_2$ and v_1 is not adjacent to v_2. Let s_1 be adjacent to u_1 in G. Then u is not adjacent to v and $u = (u_1, v_1), (s_1, v_2), (u_1, v_2) = v$ is a $u - v$ path of length 2. Thus $d_{G[H]}(u, v) = 2$. Hence the Lemma. □

Theorem 3.3. *Let G be a graph with D-matrix D_G and H, an r-regular graph with an adjacency matrix A. Let $spec_D(G) = \{\mu_1, \mu_2, \ldots, \mu_p\}$ and the ordinary spectrum of H be $\{r, \lambda_2, \lambda_3, \ldots, \lambda_n\}$. Then*

$$spec_D G[H] = \begin{pmatrix} n\mu_i + 2n - r - 2 & -(\lambda_j + 2) \\ 1 & p \end{pmatrix}, \; i = 1 \text{ to } p \text{ and } j = 2 \text{ to } n - 1$$

Proof. Using Lemma 3.2 and by a suitable ordering of vertices of $G[H]$, its D-matrix F, can be written in the form

$$F = \left[\begin{array}{cccccccccccc}
 & & & d_{12} & \cdots & d_{12} & d_{13} & \cdots & d_{13} & \cdots & \cdots & d_{1p} \\
 & A+2\overline{A} & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & d_{12} & d_{12} & d_{12} & d_{13} & \cdots & d_{13} & \cdots & \cdots & d_{1p} \\
d_{21} & \cdots & d_{21} & & & & \cdots & \cdots & \cdots & d_{2p} & \cdots & d_{2p} \\
\vdots & \vdots & \vdots & & A+2\overline{A} & & & & & \vdots & \vdots & \vdots \\
d_{21} & \cdots & d_{21} & & & & & & & d_{2p} & \cdots & d_{2p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d_{p1} & \cdots & d_{p1} & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & A+2\overline{A} & \\
d_{p1} & \cdots & d_{p1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & &
\end{array}\right]$$

$$= D_G \otimes J_n + I_p \otimes (A + 2\overline{A})$$

where $\overline{A}$ denote the adjacency matrix of $\overline{G}$.

Since H is r-regular, the all one column vector $\mathbf{1}$ of order $n \times 1$ is an eigenvector of A with an eigenvalue r. Then by Lemma 1.1, the all one vector $\mathbf{1}$ is an eigenvector of $A + 2\overline{A}$ with an eigenvalue $2n - r - 2$. Similarly if λ_j is any other eigenvalue of A with eigenvector Y_j, then Y_j is an eigenvector of $A + 2\overline{A}$ with eigenvalue $-(\lambda_j + 2)$ and that Y_j is orthogonal to $\mathbf{1}$.

Let $X_i = \begin{bmatrix} x_1^i & x_2^i & \ldots & x_p^i \end{bmatrix}^T$ be an eigenvector corresponding to the eigenvalue μ_i of D_G. Therefore

$$D_G \cdot X_i = \mu_i X_i$$

Now

$$\begin{aligned}
F \cdot (X_i \otimes \mathbf{1}_n) &= \left(D_G \otimes J_n + I_p \otimes \left(A + 2\overline{A}\right)\right)(X_i \otimes \mathbf{1}_n) \\
&= (D_G \cdot X_i) \otimes (J_n \cdot \mathbf{1}_n) + (I_p \cdot X_i) \otimes \left(A + 2\overline{A}\right) \cdot \mathbf{1}_n \\
&= \mu_i X_i \otimes n\mathbf{1}_n + X_i \otimes (2n - r - 2)\,\mathbf{1}_n \\
&= n\mu_i (X_i \otimes \mathbf{1}_n) + (2n - r - 2)(X_i \otimes \mathbf{1}_n) \\
&= (n\mu_i + 2n - r - 2)(X_i \otimes \mathbf{1}_n)
\end{aligned}$$

Therefore $n\mu_i + 2n - r - 2$ is an eigenvalue of F with eigenvector $X_i \otimes \mathbf{1}_n$.
As Y_j is orthogonal to $\mathbf{1}$, we have $J_n \cdot Y_j = 0$ for each $j = 2, 3, \ldots, n$.

Let $\{Z_k\}, k = 1, 2, \ldots, p$ be the family of p linearly independent eigenvectors associated with the eigenvalue 1 of I_p. Then for each $j = 2, 3, \ldots, n$, the p vectors $Z_k \otimes Y_j$ are eigenvectors of F with eigenvalue $-(\lambda_j + 2)$. For

$$\begin{aligned}
F \cdot (Z_k \otimes Y_j) &= \left(D_G \otimes J_n + I_p \otimes \left(A + 2\overline{A}\right)\right)(Z_k \otimes Y_j) \\
&= (D_G \cdot Z_k) \otimes (J_n \cdot Y_j) + (I_p \cdot Z_k) \otimes \left(A + 2\overline{A}\right) \cdot Y_j \\
&= 0 + Z_k \otimes -(\lambda_j + 2)\, Y_j \\
&= -(\lambda_j + 2) \cdot (Z_k \otimes Y_j)
\end{aligned}$$

Also the pn vectors $X_i \otimes \mathbf{1}_n$ and $Z_k \otimes Y_j$ are linearly independent. As the eigenvectors belonging to different eigenvalues are linearly independent and as F has a basis consisting entirely of eigenvectors, the theorem follows. $\square$

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References

[1] K. Balasubramanian, Computer Generation of Distance Polynomials of Graphs, *Journal of Computational Chemistry* **11** (1990), 829–836.

[2] F. Buckley and F. Harary, *Distance in Graphs*, Addison Wesley, 1990.

[3] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs. Theory and Applications*, Academic Press, New York – London, 1980.

[4] M. Edelberg, M. R. Garey and R. L. Graham, On the distance matrix of a tree, *Discrete Math.* **14** (1976), 230–39.

[5] P. W. Fowler, G. Caporossi and P. Hansen, Distance matrices, Wiener indices, and related invariants of fullerenes, *J. Phys. Chem. A.* **105** (2001), 6232–6242.

[6] R. L. Graham and L. Lovasz, Distance matrix polynomials of trees, *Adv. Math.* **29** (1978), 60–88.

[7] R. L. Graham and H. O. Pollak, On the addressing problem for loop switching, *Bell System Tech. J.* **50** (1971), 2495–2519.

[8] R. L. Graham and H. O. Pollak, On embedding graphs in squashed cubes, in: Y. Alavi, D. R. Lick and A. T. White (eds.), *Graph theory and applications* (Lecture notes in Mathematics, vol. 303), Springer, Berlin, 1972, 99–110.

[9] G. Indulal, I. Gutman and A. Vijayakumar, On distance energy of graphs, *MATCH Commun. Math. Comput. Chem.* **60** (2008), 461–472.

[10] G. Indulal and I. Gutman, On the distance spectra of some graphs, *Math. Commun.* **13** (2008), 123–131.

[11] G. Indulal and I. Gutman, D-equienergetic self-complementary graphs, *Kragujevac J. Math.*, in press.

[12] G. Indulal, D-spectra and D-energy of the complements of iterated line graphs of regular graphs, communicated.

[13] G. Indulal, Sharp bounds on the distance spectral radius and the distance energy of graphs, *Lin. Algebra Appl.* **430** (2009), 106–113.

[14] R. Laskar, Eigenvalues of the adjacency matrix of cubic lattice graphs, *Pacific J. Math.* **29** (1969), 623–629.

[15] M. Marcus and H. Mine, *A survey of Matrix theory and Matrix inequalities*, Allyn and Bacon, Inc., Boston, Mass., 1964.

[16] B. D. McKay, On the spectral characterization of trees, *Ars Combin.* **3** (1977), 219–232.

[17] D. Stevanović, Distance Regularity of Compositions of Graphs, *Appl. Math. Lett.* **17** (2004), 337–343.

[18] D. Stevanović and G. Indulal, The distance spectrum and energy of the compositions of regular graphs, *Appl. Math. Lett.* **22** (2009), 1136–1140.

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The distance spectrum and energy of the compositions of regular graphs[☆]

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ABSTRACT

The distance energy of a graph G is a recently developed energy-type invariant, defined as the absolute deviation of the eigenvalues of the distance matrix of G. It is a useful molecular descriptor in QSPR modelling, as demonstrated by Consonni and Todeschini in [V. Consonni, R. Todeschini, New spectral indices for molecule description, MATCH Commun. Math. Comput. Chem. 60 (2008) 3–14]. We describe here the distance spectrum and energy of the join-based compositions of regular graphs in terms of their adjacency spectrum. These results are used to show that there exist a number of families of sets of noncospectral graphs with equal distance energy, such that for any $n \in \mathbf{N}$, each family contains a set with at least n graphs. The simplest such family consists of sets of complete bipartite graphs.

1. Introduction

Let G be a simple graph on n vertices and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of its adjacency matrix A. The energy of a graph

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

was defined by Gutman in [1] and it has long known chemical applications; for details see the surveys [2–4]. Following the recent definition of the Laplacian energy in [5], it was observed that other energy-type invariants can be defined as the *absolute deviation of eigenvalues from their average value* for a suitable graph matrix. For example, let D be the distance matrix of G, indexed by the vertices of G, where D_{uv} represents the length of the shortest path between u and v in G. Then:

Definition 1 (*[6,7]*). The *distance energy* $DE(G)$ of a graph G is the sum of absolute values of the eigenvalues of the distance matrix of G.

Several invariants of this type (as well as a few others) were studied by Consonni and Todeschini [6] for possible use in QSPR modelling. Their study showed, among other things, that the distance energy is a useful molecular descriptor, since the values $DE(G)$ or $DE(G)/n$ appear among the best univariate models for the motor octane number of the octane isomers and for the water solubility of polychlorobiphenyls.

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Our motivation for this research came from an initial computer search for the pairs of graphs having equal distance energy. Since the distance energy is calculated from the distance spectrum, graphs with the same distance spectrum trivially have the same distance energy. To avoid trivial cases, we say that the graphs G and H of the same order are *DE-equienergetic* if $DE(G) = DE(H)$, while they have distinct spectra of distance matrices. Some examples of DE-equienergetic graphs are found in the literature [7–9].

The *join* $G\nabla H$ of two vertex-disjoint graphs G and H is the graph obtained from the union $G \cup H$ by adding all edges between a vertex of G and a vertex of H. Our main result (Section 2) is the description of the distance spectrum and the distance energy of the join of regular graphs in terms of their adjacency spectrum. This description is then used to show that there exist a number of families of sets of DE-equienergetic graphs, such that for any $n \in \mathbf{N}$, each family contains a set with at least n graphs. The simplest such family consists of sets of complete bipartite graphs. In Section 3 we further derive the distance spectrum of the join of a regular graph with the union of two regular graphs of distinct vertex degrees, and provide further families of sets of DE-equienergetic graphs.

2. Join of regular graphs

Theorem 2. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues of the adjacency matrix A_{G_i}, $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$. The distance spectrum of $G_1\nabla G_2$ consists of eigenvalues $-\lambda_{i,j} - 2$ for $i = 1, 2$ and $j = 2, 3, \ldots, n_i$ and two more eigenvalues of the form*

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{\left(n_1 - n_2 - \frac{r_1 - r_2}{2}\right)^2 + n_1 n_2}. \tag{1}$$

Proof. The distance matrix D of the join $G_1\nabla G_2$ has the form

$$D = \begin{bmatrix} 2(J-I) - A_{G_1} & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & 2(J-I) - A_{G_2} \end{bmatrix}.$$

As a regular graph, G_1 has the all-one vector j as an eigenvector corresponding to eigenvalue r_1, while all other eigenvectors are orthogonal to j. (Note that G_1 need not be connected, and thus, r_1 need not be a simple eigenvalue of G_1.)

Let λ be an arbitrary eigenvalue of the adjacency matrix of G_1 with corresponding eigenvector x, such that $j^{\mathrm{T}} x = 0$. Then $\left(x\; 0_{n_2 \times 1}\right)^{\top}$ is the eigenvector of D corresponding to eigenvalue $-\lambda - 2$. A similar argument holds for an arbitrary eigenvalue μ of A_{G_2}, with the corresponding eigenvector y such that $j^{\mathrm{T}} y = 0$. In this way, forming the eigenvectors of the forms $(x\; 0)^{\top}$ and $(0\; y)^{\top}$, we can construct a total of $n_1 + n_2 - 2$ mutually orthogonal eigenvectors of D. All of these eigenvectors are orthogonal to the vectors $(j\; 0)^{\top}$ and $(0\; j)^{\top}$, which means that they are spanned by the remaining two eigenvectors of D. This implies that the two remaining eigenvectors of D have the form $(\alpha j\; \beta j)^{\top}$ for a suitable choice of α and β.

Suppose now that ν is an eigenvalue of D with an eigenvector of the form $(\alpha j\; \beta j)^{\top}$. Then, from $D\,(\alpha j\; \beta j)^{\top} = \nu\,(\alpha j\; \beta j)^{\top}$, using $A_{G_1} j = r_1 j$ and $A_{G_2} j = r_2 j$, we get the system

$$\begin{aligned} (2n_1 - r_1 - 2)\alpha + n_2\beta &= \nu\alpha, \\ n_1\alpha + (2n_2 - r_2 - 2)\beta &= \nu\beta. \end{aligned}$$

Eliminating α and β we get the quadratic equation in ν

$$\nu^2 - \nu\left((2n_1 - r_1 - 2) + (2n_2 - r_2 - 2)\right) + (2n_1 - r_1 - 2)(2n_2 - r_2 - 2) - n_1 n_2 = 0,$$

whose solutions are given by (1). One easily checks that these two solutions are indeed the remaining two eigenvalues of D. $\square$

Note that the complete bipartite graph $K_{m,n}$ is isomorphic to a join $\bar{K}_m \nabla \bar{K}_n$ of the empty graphs $\bar{K}_m$ and $\bar{K}_n$. Hence,

Corollary 3. *The distance spectrum of the complete bipartite graph $K_{m,n}$ consists of simple eigenvalues $m + n - 2 \pm \sqrt{m^2 - mn + n^2}$ and an eigenvalue -2 with multiplicity $m + n - 2$.*

If $m, n \geq 2$, then $m + n - 2 \geq \sqrt{m^2 - mn + n^2}$ and we get

Corollary 4. *$DE(K_{m,n}) = 4(m + n - 2)$ for $m, n \geq 2$.*

So, any two complete bipartite graphs with the same number of vertices, apart from stars, have the same distance energy. Since the distance eigenvalues different from -2 uniquely determine parameters m and n, different complete bipartite graphs have different distance spectra. Thus, our simplest family of sets of DE-equienergetic graphs is given by

$$\{\{K_{2,n-2}, K_{3,n-3}, \ldots, K_{\lfloor n/2\rfloor, \lceil n/2\rceil}\} : n \geq 4\}.$$

The key to the successful application of Theorem 2 lies in regular graphs for which most (if not all) adjacency eigenvalues are at least -2, and so the corresponding eigenvalue $-\lambda - 2$ of the distance matrix is always negative. Such graphs are, for example, the empty graph $\bar{K}_m$, the complete graph K_m, the complete bipartite graph $K_{m/2,m/2}$ for even m, the cycle C_m, as well as regular line graphs [10] (which are themselves line graphs of regular or semiregular graphs). For such graphs, we can use the well-known fact that the sum of all adjacency eigenvalues is zero (see, e.g., [10]) in order to determine the distance energy of their join.

Theorem 5. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices, whose smallest eigenvalue of the adjacency matrix is at least -2 and such that $G_i \ncong K_n$. Then*

$$DE(G_1 \nabla G_2) = 4(n_1 + n_2) - 2(r_1 + r_2) - 8.$$

Proof. For $i = 1, 2$, denote the eigenvalues of the adjacency matrix A_{G_i} by $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$. According to Theorem 2, the distance eigenvalues of $G_1 \nabla G_2$ are

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{\left(n_1 - n_2 - \frac{r_1 - r_2}{2}\right)^2 + n_1 n_2}. \tag{2}$$

and $-\lambda_{i,j} - 2$ for $i = 1, 2$ and $j = 2, 3, \ldots, n_i$. The eigenvalues given by (2) are both nonnegative: since G_1 and G_2 are not complete, we have $n_1 \geq r_1 + 2$ and $n_2 \geq r_2 + 2$, and so

$$(2n_1 - r_1 - 2)(2n_2 - r_2 - 2) \geq n_1 n_2.$$

Adding $((n_1 - r_1/2) - (n_2 - r_2/2))^2$ to both sides, we get

$$((n_1 - r_1/2) + (n_2 - r_2/2) - 2)^2 \geq ((n_1 - r_1/2) - (n_2 - r_2/2))^2 + n_1 n_2,$$

i.e., $n_1 + n_2 - 2 - \frac{r_1+r_2}{2} - \sqrt{\left(n_1 - n_2 - \frac{r_1-r_2}{2}\right)^2 + n_1 n_2} \geq 0$. Thus, the sum of absolute values of eigenvalues (2) is equal to $(2n_1 - r_1 - 2) + (2n_2 - r_2 - 2)$.

For the remaining eigenvalues of $G_1 \nabla G_2$, from $\lambda_{i,j} \geq -2$ we have that $|-\lambda_{i,j} - 2| = \lambda_{i,j} + 2$ and therefore,

$$\begin{aligned}\sum_{j=2}^{n_1} |-\lambda_{1,j} - 2| + \sum_{j=2}^{n_1} |-\lambda_{2,j} - 2| &= \left(\sum_{j=2}^{n_1} \lambda_{1,j}\right) + 2(n_1 - 1) + \left(\sum_{j=2}^{n_2} \lambda_{2,j}\right) + 2(n_2 - 1) \\ &= -r_1 + 2(n_1 - 1) - r_2 + 2(n_2 - 1).\end{aligned}$$

We conclude that the distance energy of $G_1 \nabla G_2$ is $2(2n_1 - r_1 - 2) + 2(2n_2 - r_2 - 2)$. □

This result can be used to find new families of equienergetic graphs easily. For example, for constant sum $m + n$ we have the following sets of DE-equienergetic graphs:

$$\begin{aligned}&DE\left(\bar{K}_m \nabla \frac{n}{2} K_2\right) = 4(m+n) - 10 \quad \text{for even } n,\\ &DE\left(\bar{K}_m \nabla C_n\right) = 4(m+n) - 12,\\ &DE\left(\frac{m}{2} K_2 \nabla \frac{n}{2} K_2\right) = 4(m+n) - 12, \quad \text{for even } m \text{ and } n,\\ &DE\left(\frac{m}{2} K_2 \nabla C_n\right) = 4(m+n) - 14, \quad \text{for even } m,\\ &DE\left(C_m \nabla C_n\right) = 4(m+n) - 16,\end{aligned}$$

3. The join of a regular graph with the union of regular graphs

A computer search for pairs of DE-equienergetic graphs revealed that, among others, the wheel $W_9 \cong K_1 \nabla C_8$ and $K_1 \nabla (C_5 \cup K_3)$, which are DE-equienergetic by Theorem 5, are also DE-equienergetic to $K_1 \nabla (C_4 \cup K_4)$. However, $C_4 \cup K_4$ is not regular, but rather a union of regular graphs. Motivated by this example, we consider the distance spectrum of the graph $G_0 \nabla (G_1 \cup G_2)$, where G_0, G_1 and G_2 are regular graphs. If G_1 and G_2 have equal vertex degrees, then the distance spectrum of $G_0 \nabla (G_1 \cup G_2)$ is given by Theorem 2. Thus, we consider the case when G_1 and G_2 have distinct vertex degrees only.

Theorem 6. *For $i = 0, 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$ of the adjacency matrix A_{G_i}. If $r_1 \neq r_2$, then the distance spectrum of $G_0 \nabla (G_1 \cup G_2)$ consists of eigenvalues $-\lambda_{i,j} - 2$ for $i = 0, 1, 2$ and $j = 2, 3, \ldots, n_i$ and three more eigenvalues which are solutions of the cubic equation in ν:*

$$(2n_0 - r_0 - 2 - \nu)(\nu + r_1 + 2)(\nu + r_2 + 2) + [2(\nu + r_0 + 2) - 3n_0][n_1(\nu + r_2 + 2) + n_2(\nu + r_1 + 2)] = 0. \tag{3}$$

Proof. The distance matrix of $G_0 \nabla (G_1 \cup G_2)$ has the form

$$D = \begin{bmatrix} 2(J - I) - A_{G_0} & J & J \\ J & 2(J - I) - A_{G_1} & 2J \\ J & 2J & 2(J - I) - A_{G_2} \end{bmatrix}.$$

By analogy to the proof of Theorem 2, to every eigenvalue λ of A_{G_i} with corresponding eigenvector x, such that $j^{\top}x = 0$, there corresponds an eigenvalue $-\lambda - 2$ of D with eigenvector of D obtained by putting vector x at the coordinates corresponding to G_i and zeros at the remaining coordinates. The $n_0 + n_1 + n_2 - 3$ eigenvectors so obtained are mutually orthogonal, and also orthogonal to the vectors $\left(j\, 0_{n_1\times 1}\, 0_{n_2\times 1}\right)^{\top}$, $\left(0_{n_0\times 1}\, j\, 0_{n_2\times 1}\right)^{\top}$, $\left(0_{n_0\times 1}\, 0_{n_1\times 1}\, j\right)^{\top}$. Thus, the three remaining eigenvectors of D have the form $(\alpha j\ \beta j\ \gamma j)^{\top}$ for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

If ν is an eigenvalue of D with an eigenvector $(\alpha j\ \beta j\ \gamma j)^{\top}$, from $D\,(\alpha j\ \beta j\ \gamma j)^{\top} = \nu\,(\alpha j\ \beta j\ \gamma j)^{\top}$, and $A_{G_i} j = r_i j$ for $i = 0, 1, 2$, we get the system

$$\alpha(2n_0 - r_0 - 2) + \beta n_1 + \gamma n_2 = \nu\alpha, \tag{4}$$

$$\alpha n_0 + \beta(2n_1 - r_1 - 2) + 2\gamma n_2 = \nu\beta, \tag{5}$$

$$\alpha n_0 + 2\beta n_1 + \gamma(2n_2 - r_2 - 2) = \nu\gamma. \tag{6}$$

Assuming $\alpha = 0$ in (4)–(6), after simplifying, leads to $(r_1 - r_2)\gamma = (r_1 - r_2)\beta = 0$, which, due to $r_1 \neq r_2$, implies that $\beta = \gamma = 0$, a contradiction.

Suppose, without loss of generality, that $\alpha = 1$. Solving for β and γ and substituting solutions back into (4) yields a cubic equation (3) whose solutions, as easily seen, represent the three remaining eigenvalues of D. □

The cubic equation (3), provided $n_0 > r_0 + 2$, $n_1 \geq r_1 + 2$ and $r_1 > r_2$, has a positive solution between 0 and $2n_0 - r_0 - 2$, and a negative solution between $-r_1 - 2$ and $-r_2 - 2$. Thus, one cannot find $|\nu_1| + |\nu_2| + |\nu_3|$ without explicitly knowing the values of ν_1, ν_2 and ν_3.

Even so, Theorem 6 can be used to provide new families of sets of DE-equienergetic graphs. The main points to observe are, firstly, that the graphs G_0, G_1 and G_2 need not be connected (the only fact used in the proofs of Theorems 2 and 6 is that these graphs have the all-one vector j as an eigenvector of adjacency matrix), and, secondly, that the solutions of (3) depend only on $n_0, r_0, n_1, r_1, n_2, r_2$, and not on the structure of G_0, G_1 and G_2. Thus, we can create a set of DE-equienergetic graphs whenever we can iterate one of the graphs, say G_1, through a set of regular graphs with fixed values of n_1 and r_1.

For example, let G be an arbitrary, but fixed, regular graph with least eigenvalue at least -2. Further, for fixed $n \in \mathbf{N}$, let $\mathcal{P}_n$ be the set of integer partitions of n into parts of size at least 3. For $P = \{p_1, \ldots, p_k\} \in \mathcal{P}_n$, we denote by $\mathcal{C}_P$ the union of cycles with sizes $p_1, \ldots, p_k$. Now, Theorem 6 implies the following:

Corollary 7. *Graphs $K_1 \nabla (\mathcal{C}_P \cup G)$, $P \in \mathcal{P}_n$, form a set of DE-equienergetic graphs.*

Proof. Let G be an r-regular graph with m vertices and eigenvalues of the adjacency matrix $\lambda_1 = r \geq \lambda_2 \geq \cdots \geq \lambda_m$. For $P \in \mathcal{P}_n$, the graph $\mathcal{C}_P$ is 2-regular with n vertices. From Theorem 6, the distance eigenvalues of $K_1 \nabla (\mathcal{C}_P \cup G)$ are the solutions of the cubic equation

$$-\nu(\nu + 4)(\nu + r + 2) + (2\nu + 1)[n(\nu + r + 2) + m(\nu + 4)] = 0, \tag{7}$$

the values $-\lambda_i - 2$ for $i = 2, 3, \ldots, m$, and the values $-2\cos\frac{\pi j}{p_i} - 2$, for $p_i \in P$, $0 \leq j \leq p_i$ and $(i, j) \neq (1, 0)$ (to exclude an eigenvalue of $\mathcal{C}_P$ corresponding to the all-one eigenvector).

Let $f(n, m, r)$ be the sum of absolute values of the three solutions of (7). From the proof of Theorem 5 we know that $\sum_{i=2}^{m} |-\lambda_i - 2| = 2m - r - 2$, while the sum of $|-2\cos\frac{\pi j}{p_i} - 2|$ for $p_i \in P$ and $j = 0, \ldots, p_i$, $(i, j) \neq (1, 0)$, is equal to $2n - 4$. Thus, $DE(K_1 \nabla (\mathcal{C}_P \cup G)) = f(n, m, r) + 2n + 2m - r - 6$, regardless of the partition $P \in \mathcal{P}_n$. □

4. Concluding remarks

We have seen that the compositions of regular graphs based on the join of graphs yield a number of families containing large sets of DE-equienergetic graphs. However, the families presented here consist of dense graphs. Among sparse graphs, it is natural to start looking among trees for examples of DE-equienergetic graphs. We were surprised to find that

There exists no pair of noncospectral DE-equienergetic trees up to 20 *vertices.*

Trees have exactly one positive distance eigenvalue [11,12]. Other classes of graphs with exactly one positive distance eigenvalue include the hypermetric graphs and the graphs of negative type [13], connected bipartite graphs that are hypercube embeddable, as well as median graphs, which are the retracts of hypercubes and can be recognized in polynomial time [14].

Hence, the distance energy of a tree is twice the unique positive distance eigenvalue. The above observation then leads to the question of whether the positive distance eigenvalue determines the whole distance spectrum of a tree. More generally, to what extent does the positive distance eigenvalue characterize a tree?

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References

[1] I. Gutman, The energy of a graph, Ber. Math. Statist. Sekt. Forschungsz. Graz 103 (1978) 1–22.
[2] I. Gutman, Total π-electron energy of benzenoid hydrocarbons, Topics Curr. Chem. 162 (1992) 29–63.

[3] I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, pp. 196–211.
[4] I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology, J. Serb. Chem. Soc. 70 (2005) 441–456.
[5] I. Gutman, B. Zhou, Laplacian energy of a graph, Linear Algebra Appl. 414 (2006) 29–37.
[6] V. Consonni, R. Todeschini, New spectral indices for molecule description, MATCH Commun. Math. Comput. Chem. 60 (2008) 3–14.
[7] G. Indulal, I. Gutman, A. Vijayakumar, On distance energy of graphs, MATCH Commun. Math. Comput. Chem. 60 (2008) 461–472.
[8] H.S. Ramane, I. Gutman, D.S. Revankar, Distance equienergetic graphs, MATCH Commun. Math. Comput. Chem. 60 (2008) 473–484.
[9] G. Indulal, I. Gutman, On the distance spectra of some graphs, Math. Commun.—Univ. of Croatia 13 (2008) 123–131.
[10] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs—Theory and Application, 3rd ed., Johann Ambrosius Barth Verlag, 1995.
[11] R.L. Graham, H.O. Pollack, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971) 2495–2519.
[12] R. Bapat, S.J. Kirkland, M. Neumann, On distance matrices and Laplacians, Linear Algebra Appl. 401 (2005) 193–209.
[13] M. Deza, M. Laurent, Geometry of Cuts and Metrics, in: Algorithms and Combinatorics, vol. 15, Springer, 1997.
[14] J. Hagauer, W. Imrich, S. Klavžar, Recognizing median graphs in subquadratic time, Theoret. Comput. Sci. 215 (1999) 123–136.

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D-EQUIENERGETIC SELF-COMPLEMENTARY GRAPHS

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Abstract. The D-eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_n\}$ of a graph G are the eigenvalues of its distance matrix D and form the D-spectrum of G denoted by $spec_D(G)$. The D-energy $E_D(G)$ of the graph G is the sum of the absolute values of its D-eigenvalues. We describe here the distance spectrum of some self-complementary graphs in the terms of their adjacency spectrum. These results are used to show that there exists D-equienergetic self-complementary graphs of order $n = 48t$ and $24(2t+1)$ for $t \geq 4$.

1. INTRODUCTION

Let G be a simple graph on n vertices and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of its adjacency matrix A. The energy of a graph is defined as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

For details on this currently much studied graph–spectral invariant see [4, 5, 6]. After the introduction of the analogous concept of Laplacian energy [7], it was recognized [1] that other energy-like invariants can be defined as well, among them the *distance energy*.

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The distance matrix $D = D(G)$ of G is defined so that its (i,j)-entry d_{ij} is equal to $d_G(v_i, v_j)$, the distance between the vertices v_i and v_j of G. The eigenvalues of $D(G)$ are said to be the D-eigenvalues of G and form the D-spectrum of G, denoted by $spec_D(G)$. Since the distance matrix is symmetric, all its eigenvalues μ_i , $i = 1, 2, \dots, n$, are real and can be labelled so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

The D-energy, $E_D(G)$, of G is then defined as

$$E_D(G) = \sum_{i=1}^{n} |\mu_i| \ . \tag{1}$$

The concept of D-energy, Eq. (1), is recently introduced [11]. This definition was motivated by the much older and nowadays extensively studied graph energy. This invariant was studied by Consonni and Todeschini [1] for possible use in QSPR modelling. Their study showed, among others, that the distance energy is a useful molecular descriptor, since the values of $E_D(G)$ or $E_D(G)/n$ appear among the best univariate models for the motor octane number of the octane isomers and for the water solubility of polychlorobiphenyls. For some recent works on D-spectrum and D-energy of graphs see [8, 9, 10, 11, 13].

Two graphs with equal D-energy are said to be D-equienergetic. D-cospectral graphs are evidently D-equienergetic. Therefore, in what follows we focus our attention to D-equienergetic non$-D$-cospectral graphs. In this paper we search for self-complementary graphs of this kind. A similar work on pairs of ordinary equienergetic self-complementary graphs is [12].

All graphs considered in this paper are simple and we follow [2] for spectral graph theoretic terminology. We shall need:

Lemma 1. [2] *Let G be an r-regular connected graph, with*

$spec(G) = \{r, \lambda_2, \ldots, \lambda_n\}$. *Then*

$$spec(L^2(G)) = \begin{pmatrix} 4r-6 & \lambda_2+3r-6 & \cdots & \lambda_n+3r-6 & 2r-6 & -2 \\ 1 & 1 & \cdots & 1 & \frac{n(r-2)}{2} & \frac{nr(r-2)}{2} \end{pmatrix} .$$

Let G be a graph. Then the following construction [3] results in a self-complementary graph $\mathcal{H}$. Recall that a graph $\mathcal{H}$ is said to be self-complementary if $\mathcal{H} \cong \overline{\mathcal{H}}$, where $\overline{\mathcal{H}}$ is the complement of $\mathcal{H}$.

Construction of $\mathcal{H}$:

Replace each of the end vertices of P_4, the path on 4 vertices, by a copy of G and each of the internal vertices by a copy of $\overline{G}$. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of P_4 are adjacent.

2. DISTANCE SPECTRUM OF $\mathcal{H}$

Theorem 1. *Let G be a connected k-regular graph on n vertices, with an adjacency matrix A and spectrum $\{k, \lambda_2, \ldots, \lambda_n\}$. Then the distance spectrum of $\mathcal{H}$ consists of $-(\lambda_i + 2)$ and $\lambda_i - 1$, $i = 2, 3, \ldots, n$, each with multiplicity 2, together with the numbers*

$$\frac{1}{2}\left[7n - 3 \pm \sqrt{(2k+1)^2 + 45n^2 - 12nk - 6n}\right]$$

and

$$-\frac{1}{2}\left[n + 3 \pm \sqrt{(2k+1)^2 + 5n^2 + 4nk + 2n}\right] .$$

Proof. Let G be a connected k-regular graph on n vertices with an adjacency matrix A and spectrum $\{k, \lambda_2, \ldots, \lambda_n\}$. Let $\mathcal{H}$ be the self-complementary graph obtained from G by the above construction. Then the distance matrix D of $\mathcal{H}$ has the form

$$\begin{bmatrix} 2(J-I)-A & J & 2J & 3J \\ J & J-I+A & J & 2J \\ 2J & J & J-I+A & J \\ 3J & 2J & J & 2(J-I)-A \end{bmatrix} .$$

As a regular graph, G has the all-one vector j as an eigenvector corresponding to eigenvalue k, while all other eigenvectors are orthogonal to j. Also corresponding to the eigenvalue $\lambda \neq k$ of G, $\overline{G}$ has the eigenvalue $-1-\lambda$ such that both λ and $-1-\lambda$ have same multiplicities and eigenvectors.

Let λ be an arbitrary eigenvalue of the adjacency matrix of G with corresponding eigenvector x, such that $j^T x = 0$. Then $\begin{pmatrix} x & 0 & 0 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 0 & 0 & x \end{pmatrix}^T$ are the eigenvectors of D corresponding to eigenvalue $-\lambda - 2$. Corresponding to an arbitrary eigenvalue λ of G , $-\lambda-2$ is an eigenvalue of D with multiplicity 2. Similarly $\begin{pmatrix} 0 & x & 0 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 0 & x & 0 \end{pmatrix}^T$ are the eigenvectors of D corresponding to the eigenvalue $\lambda - 1$.

In this way, forming eigenvectors of the form

$$\begin{pmatrix} x & 0 & 0 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & x & 0 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 0 & x & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 0 & 0 & x \end{pmatrix}^T$$

we can construct a total of $4(n-1)$ mutually orthogonal eigenvectors of D. All these eigenvectors are orthogonal to the vectors

$$\begin{pmatrix} j & 0 & 0 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & j & 0 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 0 & j & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 0 & 0 & j \end{pmatrix}^T.$$

The four remaining eigenvectors of D are of the form $\Psi = (\alpha j, \beta j, \gamma j, \delta j)^T$ for some $(\alpha, \beta, \gamma, \delta) \neq (0,0,0,0)$.

Now, suppose that ν is an eigenvalue of D with an eigenvector Ψ. Then from $D\Psi = \nu\Psi$, we get

$$\begin{aligned}
[2(n-1)-k]\,\alpha + n\beta + 2n\gamma + 3n\delta &= \nu\alpha && (2)\\
n\alpha + (n-1+k)\,\beta + n\gamma + 2n\delta &= \nu\beta && (3)\\
2n\alpha + n\beta + (n-1+k)\,\gamma + n\delta &= \nu\gamma && (4)\\
3n\alpha + 2n\beta + n\gamma + [2(n-1)-k]\,\delta &= \nu\delta\,. && (5)
\end{aligned}$$

Claim: $\alpha \neq 0$. If $\alpha = 0$, then by solving equations (3)–(5) we get $\beta = g_1\gamma$ and $\delta = g_2\gamma$ for some constants g_1 and g_2. Then using $\beta + 2\gamma + 3\delta = 0$, we obtain

$$\left[11n^2 + n\,(4k+2) + 12k^2 + 12k + 3\right]\gamma = 0$$

which implies that $\gamma = \beta = \delta = 0$, which is impossible.

Thus $\alpha \neq 0$ and without loss of generality we may set $\alpha = 1$.

Then by solving equations (3)–(5) for β, γ, and δ, and substituting these values into equation (2), we arrive at a biquadratic equation in ν:

$$\begin{aligned} & \left[\nu^2 - (7n-3)\,\nu + n\,(n+3k-9) - \left(k^2+k-2\right)\right] \\ \times \quad & \left[\nu^2 + (n+3)\,\nu - n\,(n+k-1) - \left(k^2+k-2\right)\right] = 0 \end{aligned}$$

whose solutions

$$\frac{1}{2}\left[7n - 3 \pm \sqrt{(2k+1)^2 + 45n^2 - 12nk - 6n}\right]$$

and

$$-\frac{1}{2}\left[n + 3 \pm \sqrt{(2k+1)^2 + 5n^2 + 4nk + 2n}\right]$$

as easily seen, represent the four remaining eigenvalues of D. Hence the theorem. $\square$

Corollary 1. *Let G be a connected k-regular graph on n vertices with an adjacency matrix A and spectrum $\{k, \lambda_2, \dots, \lambda_n\}$. Let $\mathcal{H}$ be the self-complementary graph obtained from G by the above described construction. Then*

$$E_D(\mathcal{H}) = 7n - 3 + \sqrt{(2k+1)^2 + 5n^2 + 4nk + 2n} + \sum_{i=2}^{n} |\lambda_i + 2| + \sum_{i=2}^{n} |\lambda_i - 1| \ .$$

3. A PAIR OF D-EQUIENERGETIC SELF-COMPLEMENTARY GRAPHS

In this section we demonstrate the existence of a pair of D-equienergetic self-complementary graphs on n vertices for $n = 48t$ and $n = 24(2t+1)$ for all $t \geq 4$. For this we first prove:

Theorem 2. *For every $n \geq 8$, there exists a pair of 4-regular non-cospectral graphs on n vertices.*

Proof. We shall consider the following two cases.

Case 1: $n = 2t$, $t \geq 4$. In this case form two t-cycles $u_1 u_2 \ldots u_t$ and $v_1 v_2 \ldots v_t$ and join u_i to v_i for each i . Let $\mathcal{A}$ be the resulting graph. Let $\mathcal{B}_1$ be the graph obtained from $\mathcal{A}$ by making u_i adjacent with v_{i+1} for each i and $\mathcal{B}_2$ be obtained by making u_i adjacent with v_{i+2} for each i where suffix addition is modulo t . Then both $\mathcal{B}_1$ and $\mathcal{B}_2$ are 4-regular and the number of triangles in $\mathcal{B}_1$ is $2t$ and that in $\mathcal{B}_2$ is zero. Thus $\mathcal{B}_1$ and $\mathcal{B}_2$ are non-cospectral.

In Figure 1 we illustrate the above construction for $t = 4$.

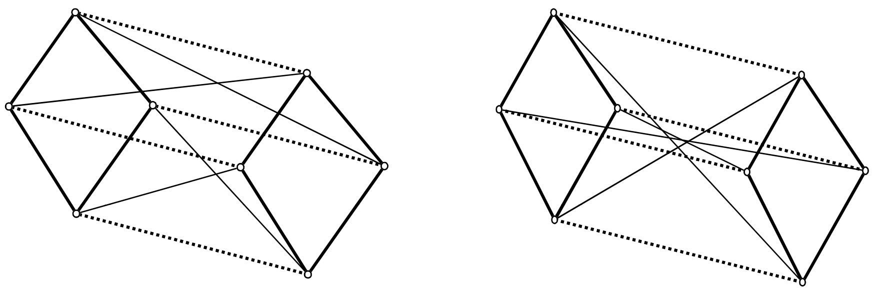

Figure 1. The graphs $\mathcal{B}_1$ and $\mathcal{B}_2$ in the case $t = 4$.

Case 2: $n = 2t + 1$, $t \geq 4$.

In this case form the $(t+1)$-cycle $v_1 v_2 \ldots v_t v_{t+1}$ and the t-cycle $u_1 u_2 \ldots u_t$. Now make v_{t-1} adjacent with v_1 and v_i with u_i , $i = 1, \ldots, t$. Then join v_j to u_{j+2} , $j = 2, \ldots, t-2$, v_t to u_2 and then v_{t+1} to u_1 and u_3 . Let $\mathcal{F}_1$ be the resulting graph. Then $\mathcal{F}_1$ is 4-regular and contains two triangles $v_1 v_2 v_3$ and $v_5 u_1 v_1$ for $t = 4$ and only one triangle $v_{t+1} u_1 v_1$ for $t \geq 5$.

To get the other 4-regular graph, form the $(2t+1)$-cycle $v_1 v_2 \ldots v_t v_{t+1} \ldots v_{2t+1}$. Join v_i to v_{i+2} , $i = 1, 3, 5, \ldots, 2t+1, 2, 4, 6, \ldots, 2t$. Let $\mathcal{F}_2$ be the resulting graph. Then it is 4-regular and contains $2t+1$ triangles. Thus the graphs $\mathcal{F}_1$ and $\mathcal{F}_2$ are not cospectral. $\square$

In Figure 2 we illustrate the above construction for $t = 4$.

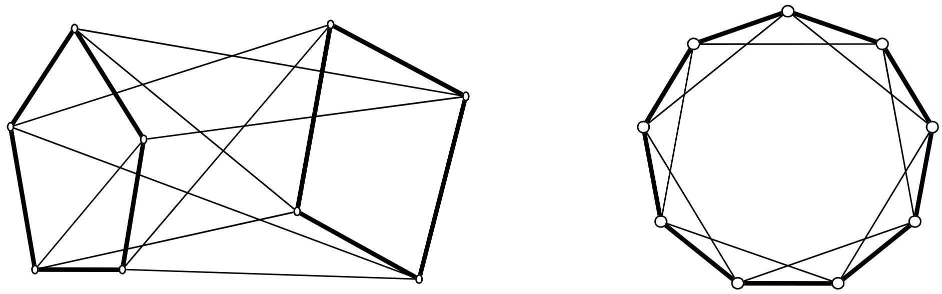

Figure 2. The graphs $\mathcal{F}_1$ and $\mathcal{F}_2$ in the case $t = 4$.

Theorem 3. *Let G be a connected 4-regular graph on n vertices, with an adjacency matrix A and spectrum $\{4, \lambda_2, \ldots, \lambda_n\}$. Let $H = L^2(G)$ and $\mathcal{H}$ be the P_4 self-complementary graph obtained from H, according to the above described construction. Then*

$$E_D(\mathcal{H}) = 3[8(3n-1) + \sqrt{20n^2 + 28n + 49}] \ .$$

Proof follows from Theorem 1, Lemma 1, and the fact that both $\lambda_i + 3r - 4$ and $\lambda_i + 3r - 7$ are positive when $r = 4$. □

Theorem 4. *For every $n = 48t$ and $n = 24(2t+1)$, $t \geq 4$, there exists a pair of D-equienergetic self-complementary graph.*

Proof. Case 1: $n = 48t$

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the two non-cospectral 4-regular graphs on $2t$ vertices as given by Theorem 2. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively denote their second iterated line graphs. Then both are on $12t$ vertices and are 6-regular. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be the respective self-complementary graphs on $48t$ vertices. Then by Theorem 3, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are D-equienergetic.

The other case $n = 24(2t+1)$ can be proven in a similar manner by considering the two non-cospectral 4-regular graphs on $2t+1$ vertices whose structure is outlined in Theorem 2. □

4. D-ENERGY OF SOME SELF-COMPLEMENTARY GRAPHS

The D-energy of some self-complementary graphs $\mathcal{H}$ is easily deduced from the adjacency spectra of the respective parent graphs G.

1. If $G \cong K_n$, the complete graph on n vertices, then

$$E_D(\mathcal{H}) = \begin{cases} 4 + 2\sqrt{10} & \text{for } n = 1 \\ 6 + 3\sqrt{17} + \sqrt{41} & \text{for } n = 2 \\ 22 + 2\sqrt{85} & \text{for } n = 3 \\ 13n - 9 + \sqrt{13n^2 - 6n + 1} & \text{for } n \geq 4 \,. \end{cases}$$

2. If $G \cong K_{p,p}$, the complete bipartite graph on $n = 2p$ vertices, then

$$E_D(\mathcal{H}) = 15n - 17 + \sqrt{8n^2 + 4n + 1} \,.$$

3. If $G \cong CP(n)$, the cocktail party graph on n vertices, then

$$E_D(\mathcal{H}) = 13n - 9 + \sqrt{13n^2 - 18n + 9} \,.$$

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References

[1] V. Consonni, R. Todeschini, *New spectral indices for molecule description*, MATCH Commun. Math. Comput. Chem. **60** (2008), 3–14.

[2] D. M. Cvetkovic, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980.

[3] A. Farrugia, *Self-complementary graphs and generalisations: A comprehensive reference manual*, M. Sc. Thesis, University of Malta, 1999.

[4] I. Gutman, *The energy of a graph: old and new results*, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, 2001, pp. 196–211.

[5] I. Gutman, X. Li, J. Zhang, *Graph energy*, in: M. Dehmer, F. Emmert–Streib (Eds.), *Analysis of Complex Networks. From Biology to Linguistics*, Wiley–VCH, Weinheim, 2009, in press.

[6] I. Gutman, O. E. Polansky, *Mathematical Concept in Organic Chemistry*, Springer–Verlag, Berlin, 1986, Chapter 8.

[7] I. Gutman, B. Zhou, *Laplacian energy of a graph*, Lin. Algebra Appl. **414** (2006), 29–37.

[8] G. Indulal, *Sharp bounds on the distance spectral radius and the distance energy of graphs*, Lin. Algebra Appl. **430** (2009), 106-113.

[9] G. Indulal, *D-spectra and D-energy of the complements of iterated line graphs of regular graphs*, communicated.

[10] G. Indulal, I. Gutman, *On the distance spectra of some graphs*, Math. Commun. **13** (2008), 123–131.

[11] G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **60** (2008), 461–472.

[12] G. Indulal, A. Vijayakumar, *Equienergetic self-complementary graphs*, Czechoslovak Math J. **58** (2008) 911–919.

[13] D. Stevanović, G. Indulal, *The distance spectrum and energy of the compositions of regular graphs*, communicated.

Bibliography

[1] R. Albert, H.Jeong, A. L. Barabasi, *Diameter of the world-wide web*, Nature, **401**(1999), 130 - 131.

[2] R. Albert, A. L. Barabasi, *Statistical mechanics of complex networks*, Rev. Mod. Phys., **74**(2002), 47 - 97.

[3] D.M.Cvetkovic, M. Doob, H. Sachs, *Spectra of Graphs-Theory and Applications*, Academic Press, (1980).

[4] A. Farrugia, *Self-complementary graphs and generalisations*:A comprehensive reference manual, M.Sc Thesis, University of Malta(1999).

[5] I. Gutman, *The energy of a graph*, Ber. Math.Statist. Sekt. Forschungsz. Graz 103 (1978), 1-22.

[6] I. Gutman, *Total π-electron energy of benzenoid hydrocarbons*, Topics Curr. Chem. 162 (1992), 29–63.

[7] D. König, *Theorie der Endlichen und Unendlichen Graphen*, Leipzig(1936).

[8] C. Berge, *The Theory of Graphs*, Methuen, (1962).

[9] O.Ore, *Theory of Graphs*, Amer. Math. Soc., Providence, (1962).

[10] F. Harary, *Graph Theory*, Narosa, (2000).

[11] N. L. Biggs, E. K. Lloyd, R. J. Wilson, *Graph Theory 1736 – 1936*, Oxford University Press, (1976).

[12] F. S. Roberts, *Discrete Mathematical Models with Application to Social, Biological and Environmental Problems*, Prentice - Hall, (1976).

[13] L. W. Beineke, R. J. Wilson, *Graph Connections*, Oxford University Press, (1997).

[14] I. Gutman, *The energy of a graph: old and new results*, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, pp. 196–211.

[15] I. Gutman, *Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology*, J. Serb. Chem. Soc. 70 (2005), 441–456.

[16] I. Gutman, B. Zhou, *Laplacian energy of a graph*, Linear Algebra Appl. 414 (2006), 29–37.

[17] V. Consonni, R. Todeschini, *New Spectral Indices for Molecule Description*, MATCH Commun. Math. Comput. Chem. 60 (2008), 3–14.

[18] G. Indulal , I. Gutman , A. Vijayakumar , *On distance energy of graphs* , MATCH Commun. Math. Comput. Chem. **60**(2008) 461 – 472. .

[19] G. Indulal, A. Vijayakumar, *Equienergetic self-complementary graphs*, Czechoslovak Math J. ,**4**(2008).

[20] G. Indulal , I. Gutman , *On the distance spectra of some graphs* , Math. Commun. ,**13**(2008) 123–131

[21] G. Indulal , *$D-$ spectra and $D-$ energy of the complements of iterated line graphs of regular graphs* (Communicated).

[22] D. Stevanović ,G. Indulal , *The distance spectrum and energy of the compositions of regular graphsAppl. Math. Lett.*, 22:136 – 1140, 2009.

[23] G. Indulal , *Sharp bounds on the distance spectral radius and the distance energy of graphs* , Linear Algebra Appl. **430** (2009), 106-113